Planar Straight-line Realizations of 2-Trees with Prescribed Edge Lengths

Carlos Alegría, Manuel Borrazzo, Giordano Da Lozzo
Giuseppe Di Battista, Fabrizio Frati, and Maurizio Patrignani

Roma Tre University
Rome, Italy

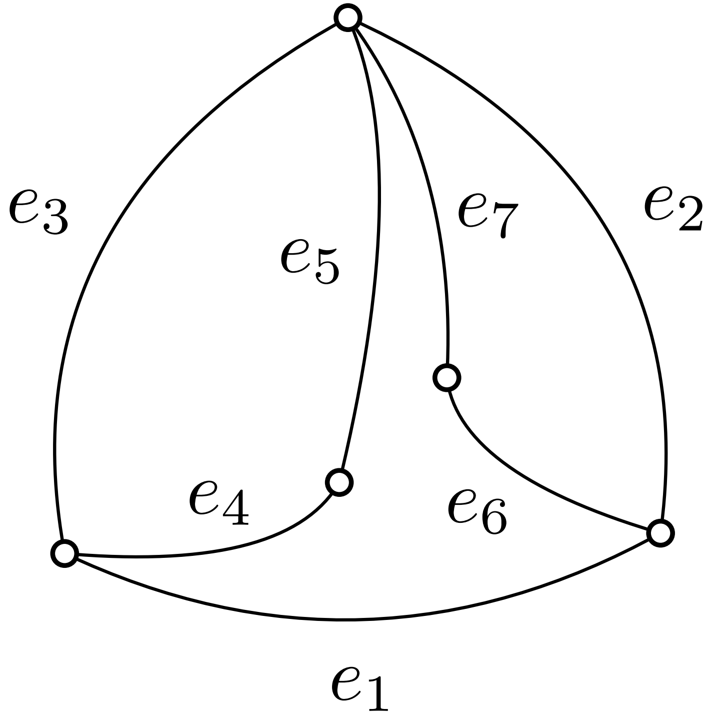
Tübingen, Germany
September 14-17

Fixed Edge-Length Planar Realization problem

Fixed Edge-Length Planar Realization problem

$$G = (V, E, \lambda)$$
$$\lambda : E \rightarrow \mathbb{R}^+$$

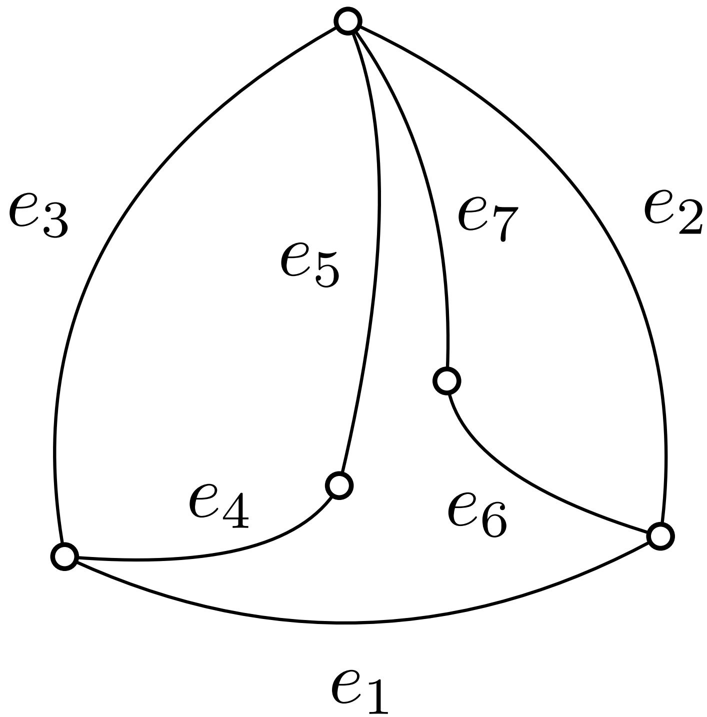
Fixed Edge-Length Planar Realization problem



$$G = (V, E, \lambda)$$

$$\lambda : E \to \mathbb{R}^+$$

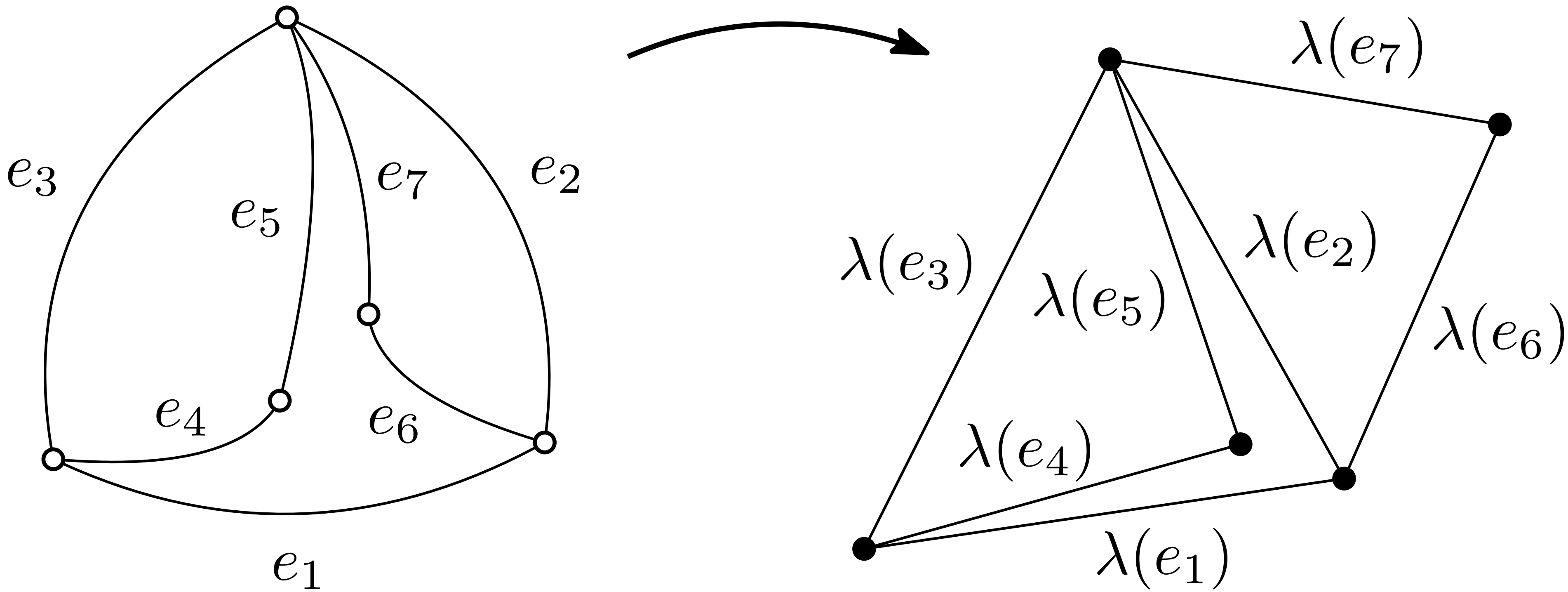
Fixed Edge-Length Planar Realization problem



$$G = (V, E, \lambda)$$
$$\lambda : E \to \mathbb{R}^+$$

$$\Gamma$$

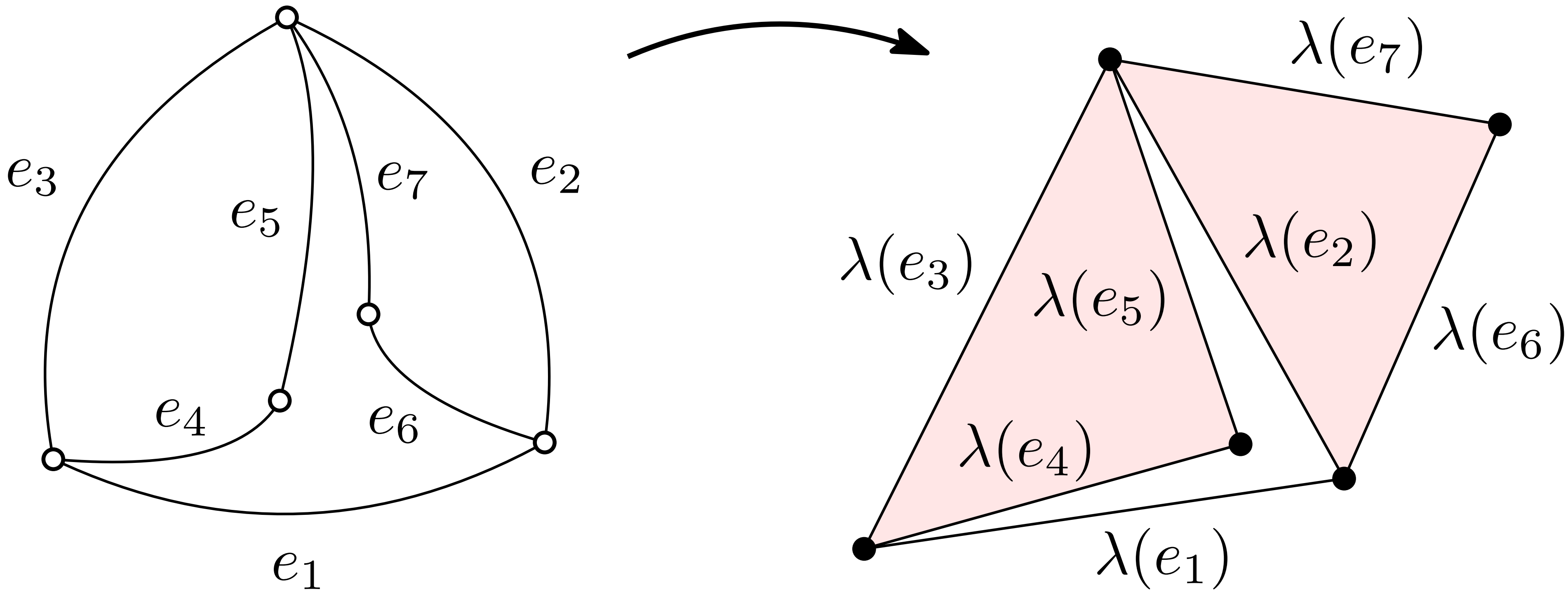
Fixed Edge-Length Planar Realization problem



$$G = (V, E, \lambda)$$
$$\lambda : E \to \mathbb{R}^+$$

$$\Gamma$$

Fixed Edge-Length Planar Realization problem



$$G = (V, E, \lambda)$$
$$\lambda : E \to \mathbb{R}^+$$

$$\Gamma$$

Fixed Edge-Length Planar Realization problem

[Eades & Wormald, 1990]

- Introduced the problem

Fixed Edge-Length Planar Realization problem

[Eades & Wormald, 1990]

- Introduced the problem
- NP-hard for:
  - Triconnected planar graphs
  - Biconnected planar graphs with unit lengths

Fixed Edge-Length Planar Realization problem

[Eades & Wormald, 1990]

- Introduced the problem
- NP-hard for:
  - Triconnected planar graphs
  - Biconnected planar graphs with unit lengths

[Cabello, Demaine, and Rote, 2007]

NP-hard for triconnected planar graphs with unit lengths

Fixed Edge-Length Planar Realization problem

[Eades & Wormald, 1990]

- Introduced the problem
- NP-hard for:
  - Triconnected planar graphs
  - Biconnected planar graphs with unit lengths

[Cabello, Demaine, and Rote, 2007]

NP-hard for triconnected planar graphs with unit lengths

[Abel et al., 2016]

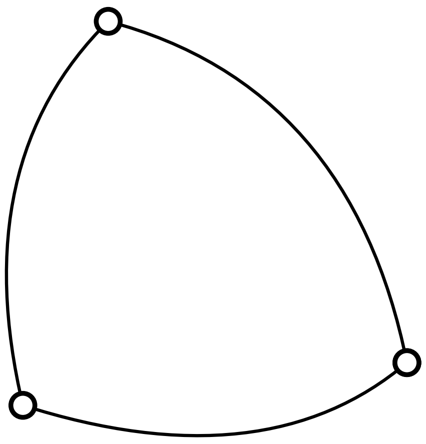
Recognizing *matchstick graphs* is $\exists\mathbb{R}$-complete

Fixed Edge-Length Planar Realization problem

2-Trees

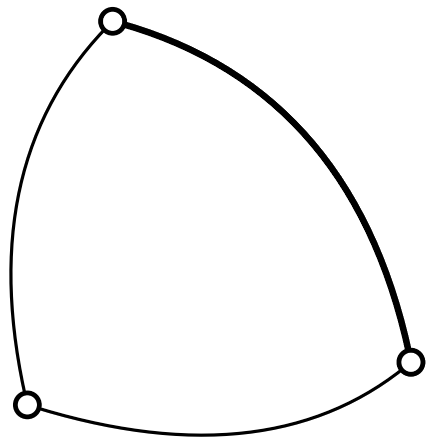
Fixed Edge-Length Planar Realization problem

2-Trees



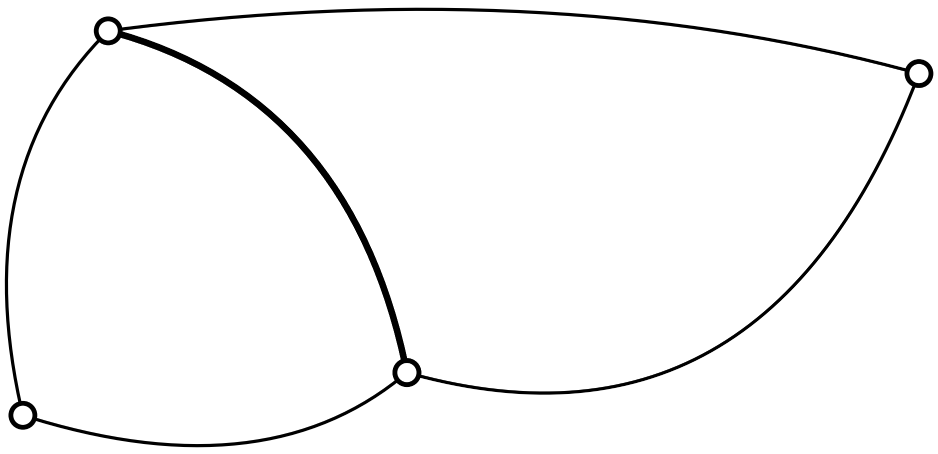
Fixed Edge-Length Planar Realization problem

2-Trees



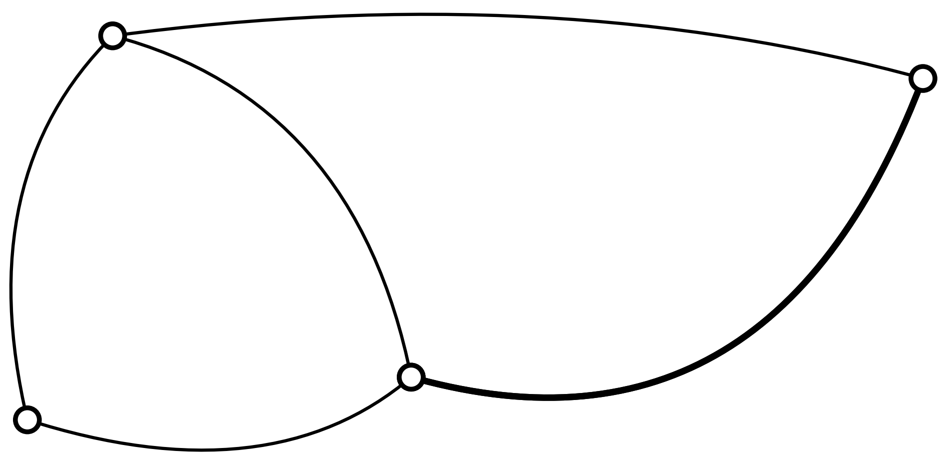
Fixed Edge-Length Planar Realization problem

2-Trees



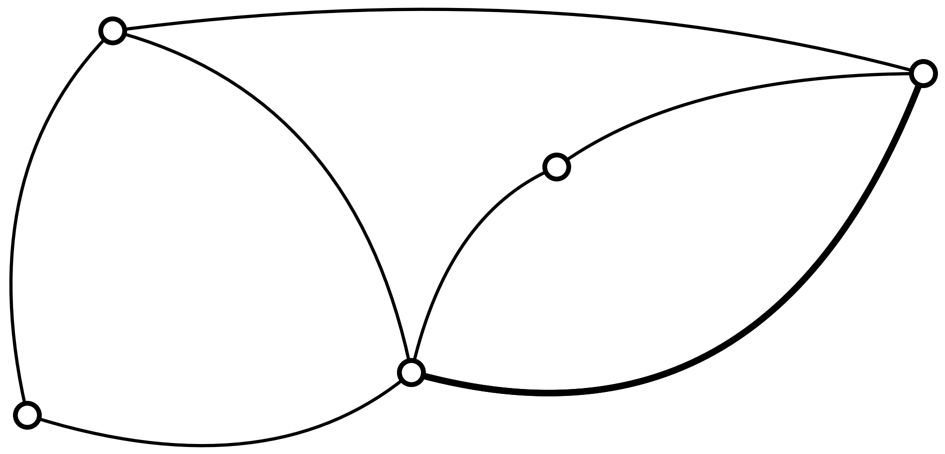
Fixed Edge-Length Planar Realization problem

2-Trees



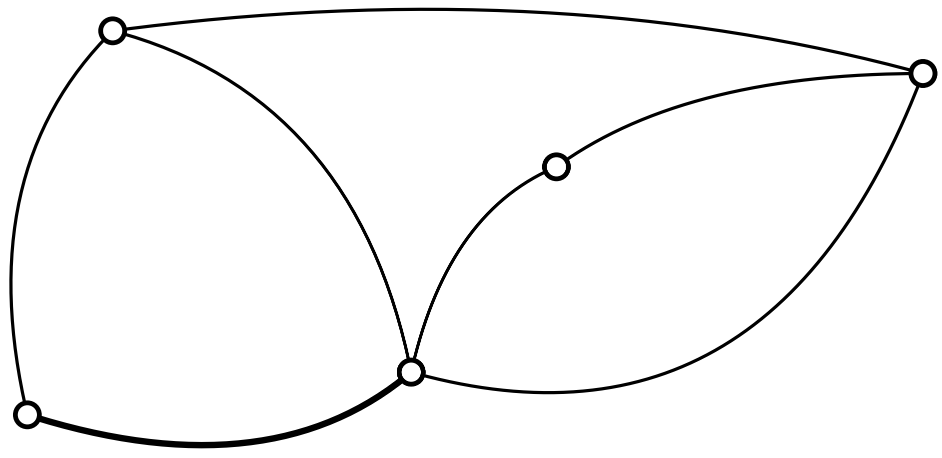
Fixed Edge-Length Planar Realization problem

2-Trees



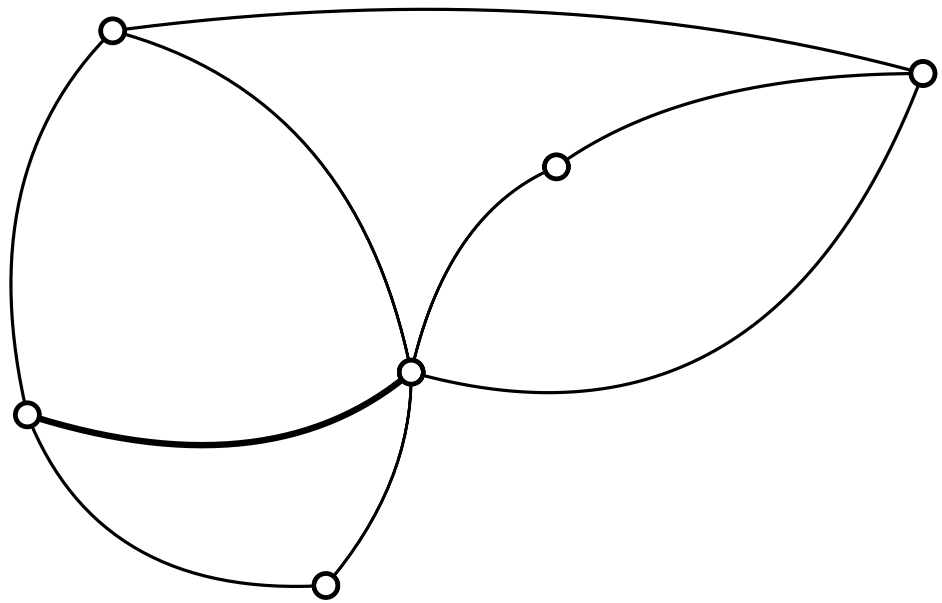
Fixed Edge-Length Planar Realization problem

2-Trees



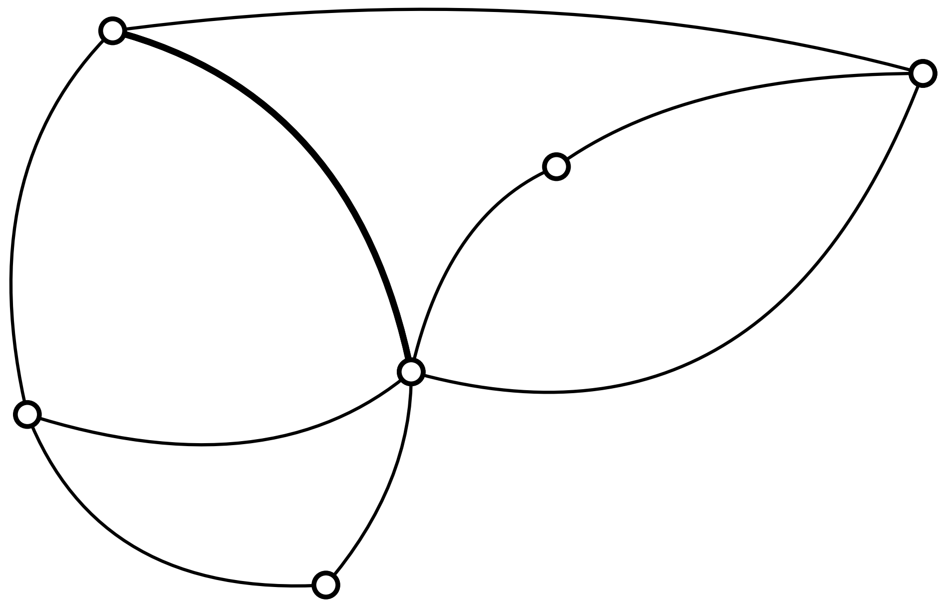
Fixed Edge-Length Planar Realization problem

2-Trees



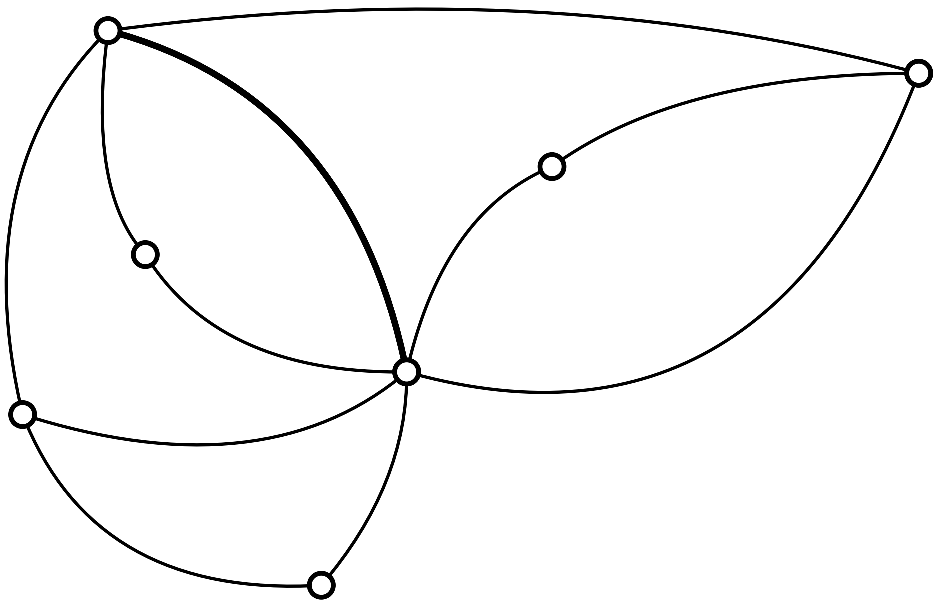
Fixed Edge-Length Planar Realization problem

2-Trees



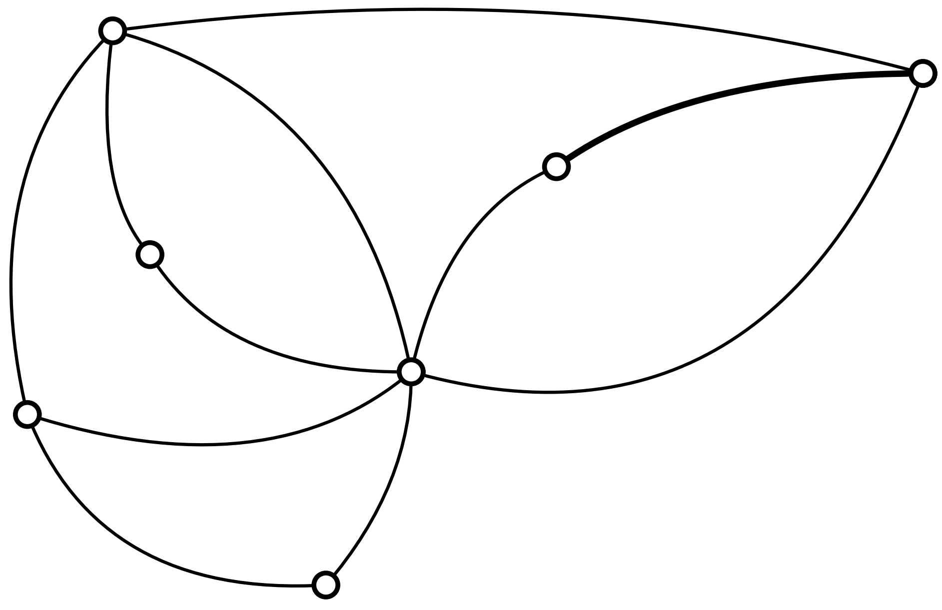
Fixed Edge-Length Planar Realization problem

2-Trees



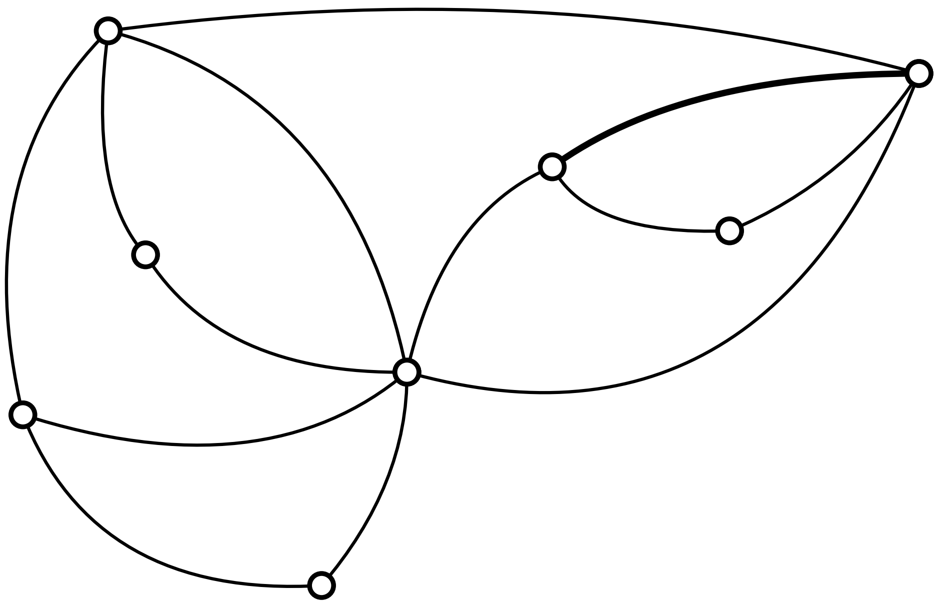
Fixed Edge-Length Planar Realization problem

2-Trees



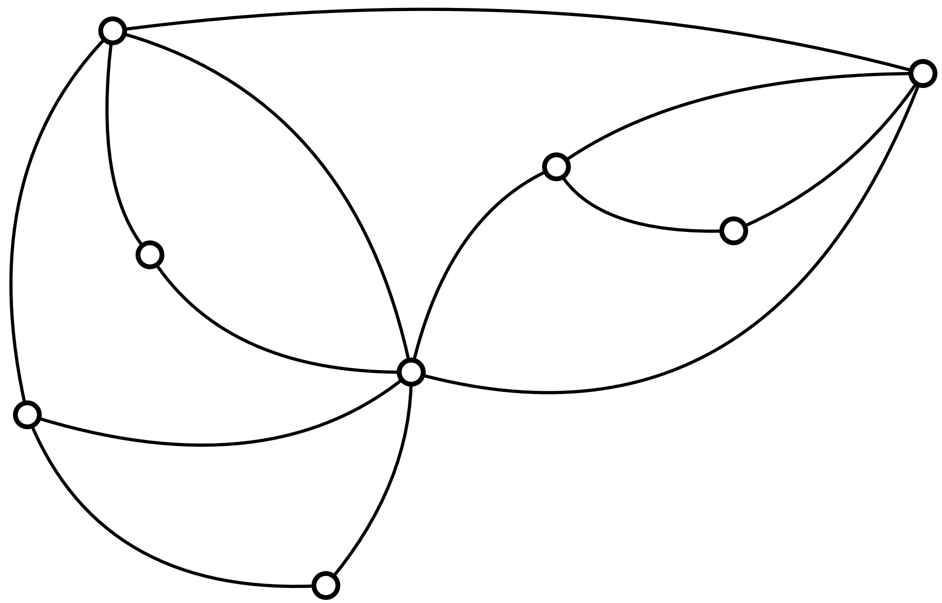
Fixed Edge-Length Planar Realization problem

2-Trees



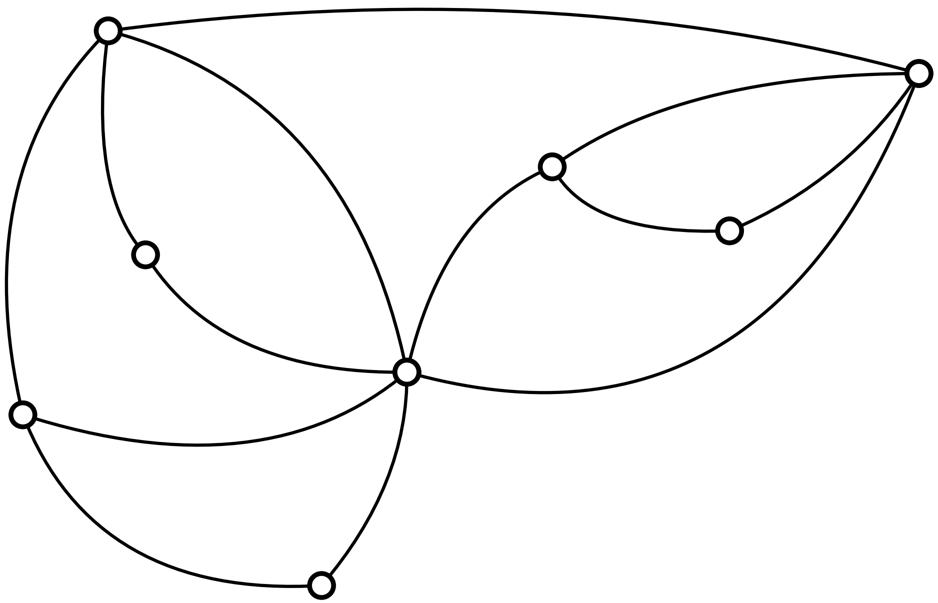
Fixed Edge-Length Planar Realization problem

2-Trees



Fixed Edge-Length Planar Realization problem

2-Trees



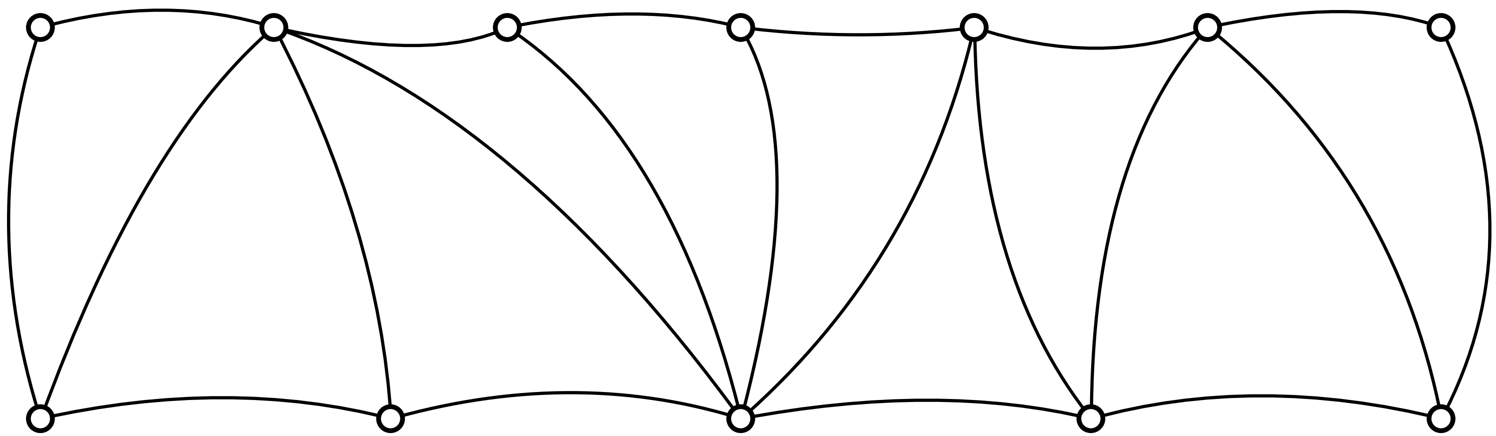
Maximal planar graphs with no K_4 minors

Fixed Edge-Length Planar Realization problem

Maximal outerplanar graphs

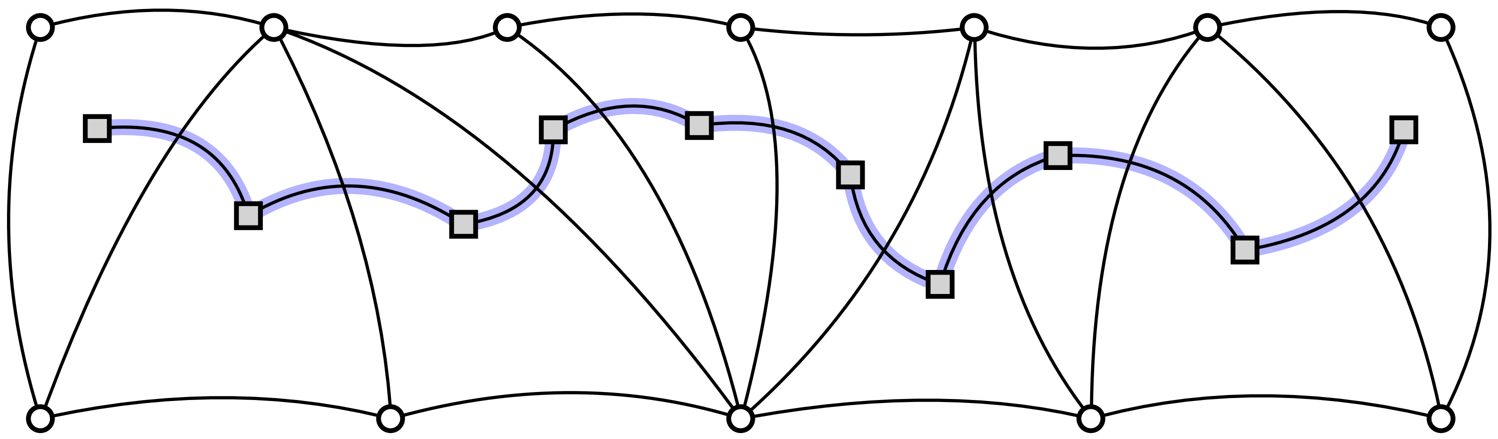
Fixed Edge-Length Planar Realization problem

Maximal outerplanar graphs



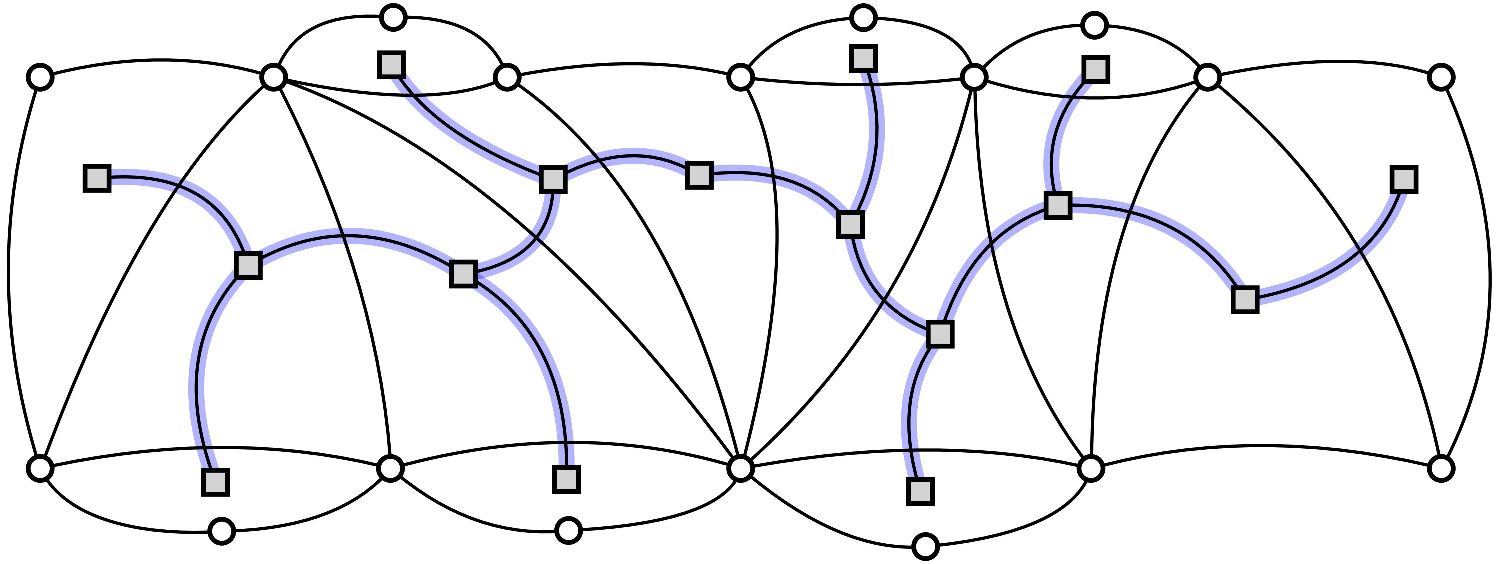
Fixed Edge-Length Planar Realization problem

Maximal outerplanar graphs



Outerpath

Fixed Edge-Length Planar Realization problem
Maximal outerplanar graphs



Outerpillar

Results

- Fixed embedding:

Results

- Fixed embedding:
    - Linear-time algorithm*

* NP-hard for general graphs.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:

* NP-hard for general graphs.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4

* NP-hard for general graphs.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2**

* NP-hard for general graphs.

** NP-hard for general graphs with 1 length.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2**
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length

* NP-hard for general graphs.
** NP-hard for general graphs with 1 length.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2**
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length
  - Linear-time algorithm for outerpaths

* NP-hard for general graphs.

** NP-hard for general graphs with 1 length.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2**
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length
  - Linear-time algorithm for outerpaths
  - Cubic-time algorithm for outerpillars

* NP-hard for general graphs.

** NP-hard for general graphs with 1 length.

Results

- Fixed embedding:
  - Linear-time algorithm*
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2**
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length
  - Linear-time algorithm for outerpaths
  - Cubic-time algorithm for outerpillars

* NP-hard for general graphs.

** NP-hard for general graphs with 1 length.

NP-hardness

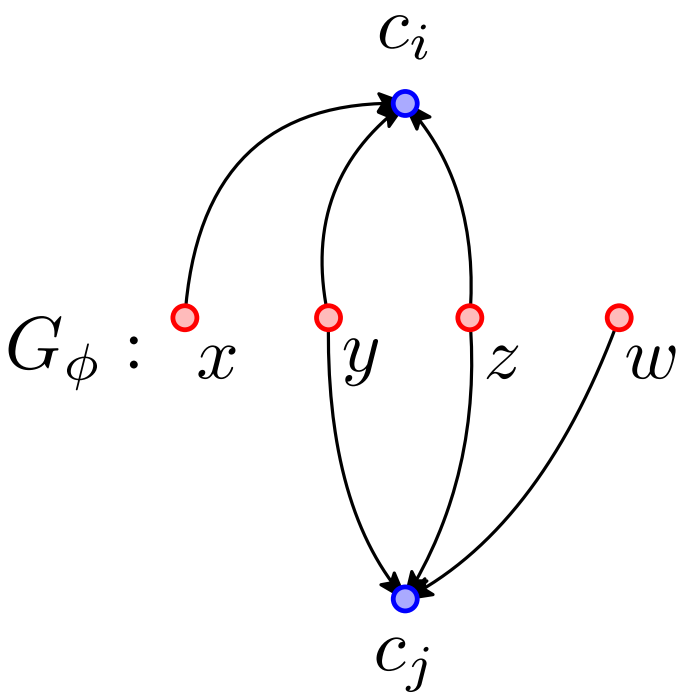
Planar Monotone 3-SAT

Planar Monotone 3-SAT

$$\phi: \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$$

Planar Monotone 3-SAT

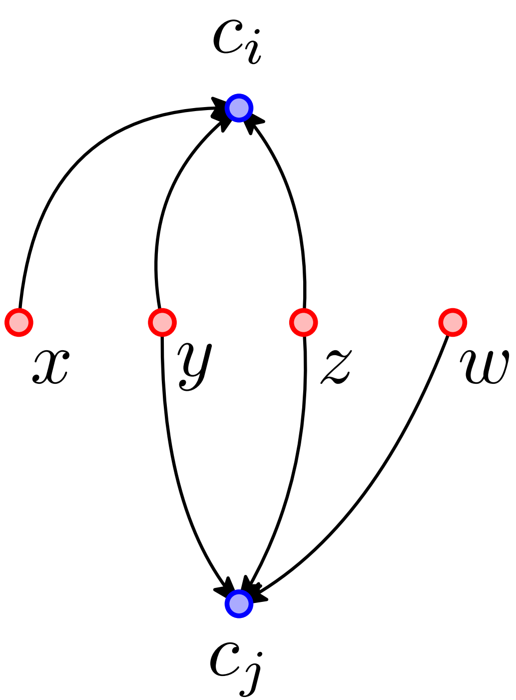
$$\phi: \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$$



Planar Monotone 3-SAT

$$\phi : \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$$

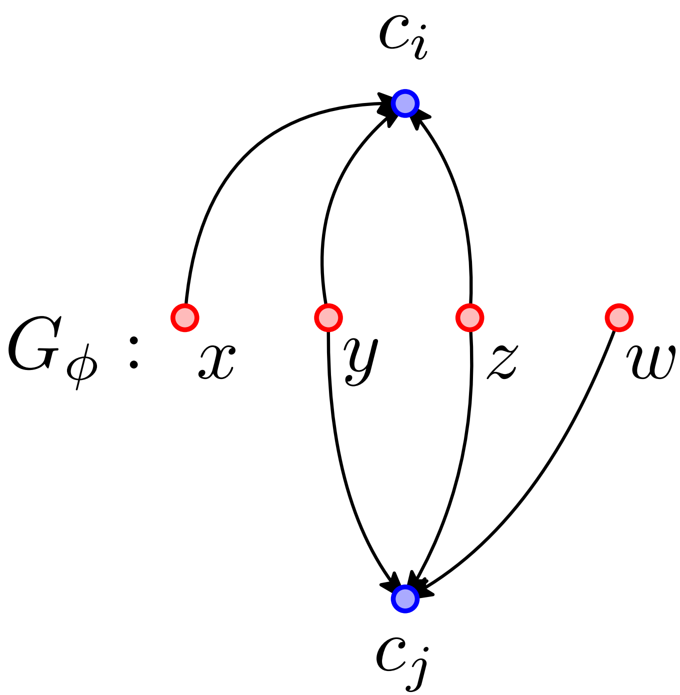
G_ϕ :



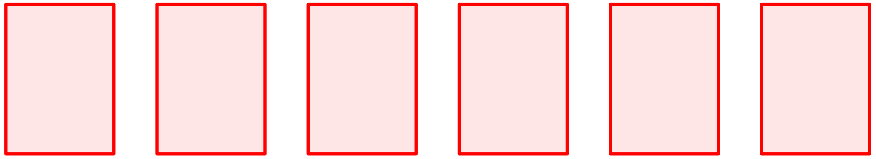
Γ_ϕ :

Planar Monotone 3-SAT

$$\phi : \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$$



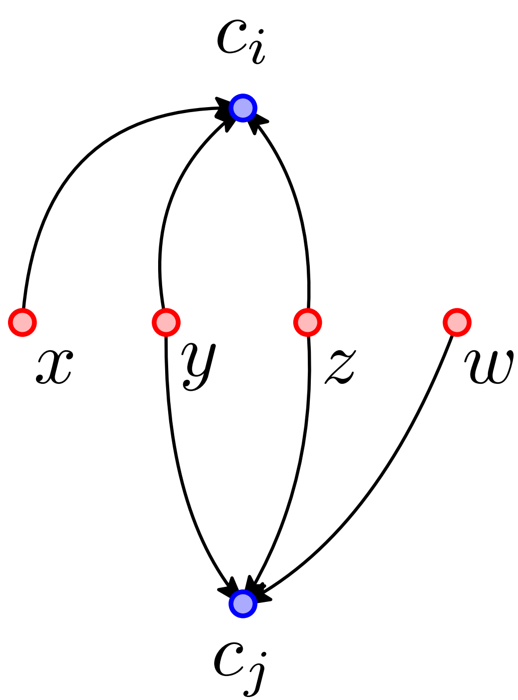
Γ_ϕ :



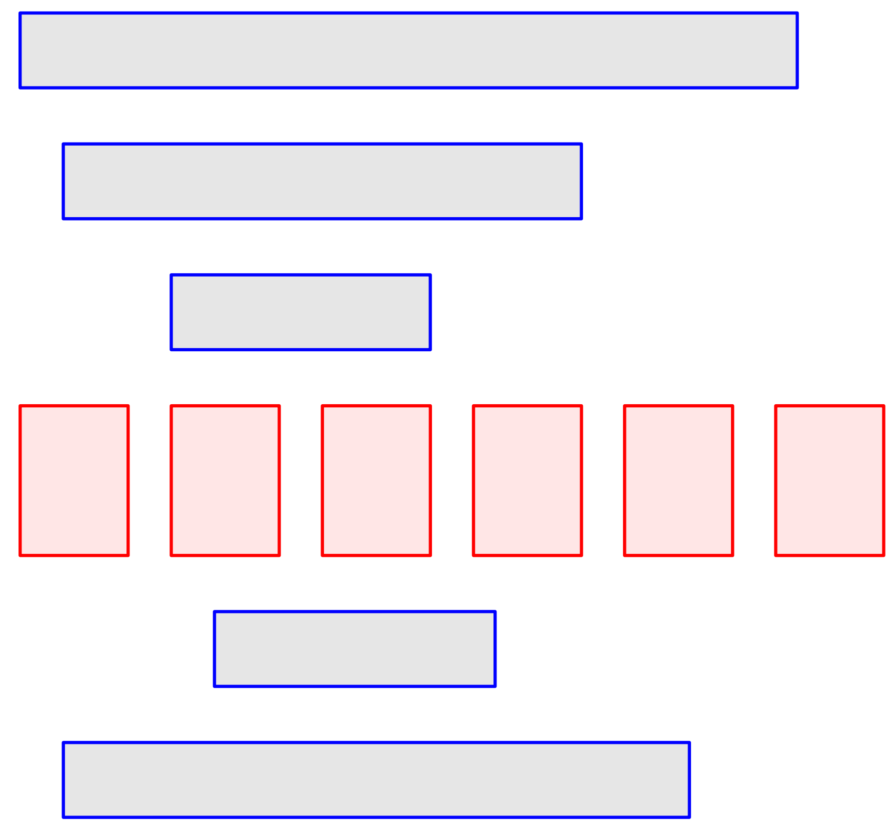
Planar Monotone 3-SAT

$$\phi : \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$$

G_ϕ :



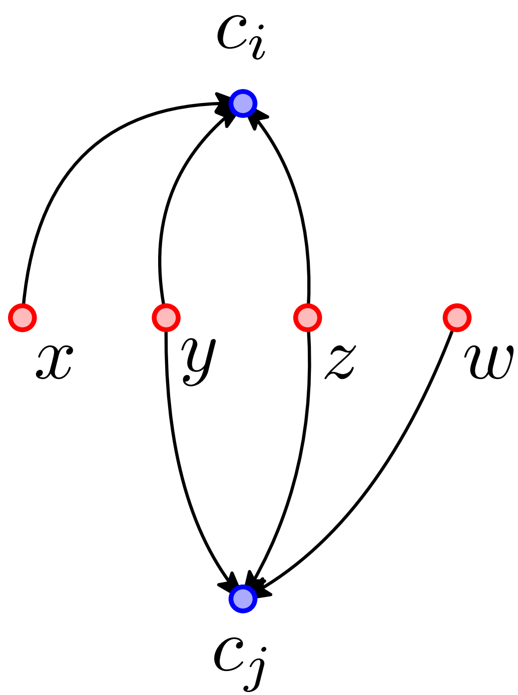
Γ_ϕ :



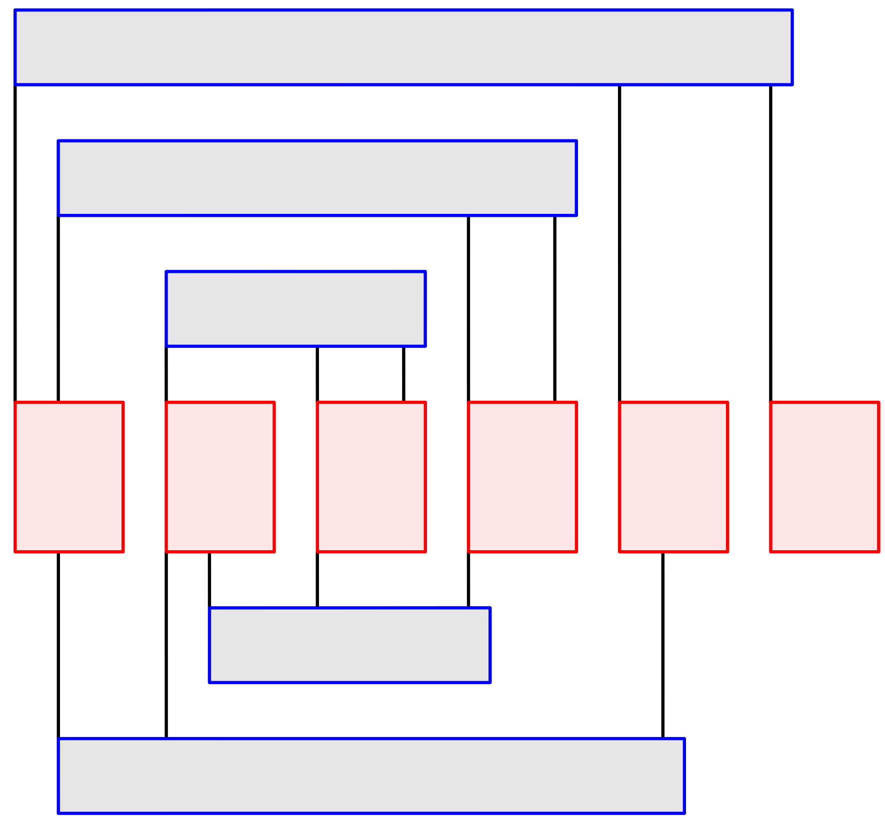
Planar Monotone 3-SAT

$$\phi : \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$$

G_ϕ :

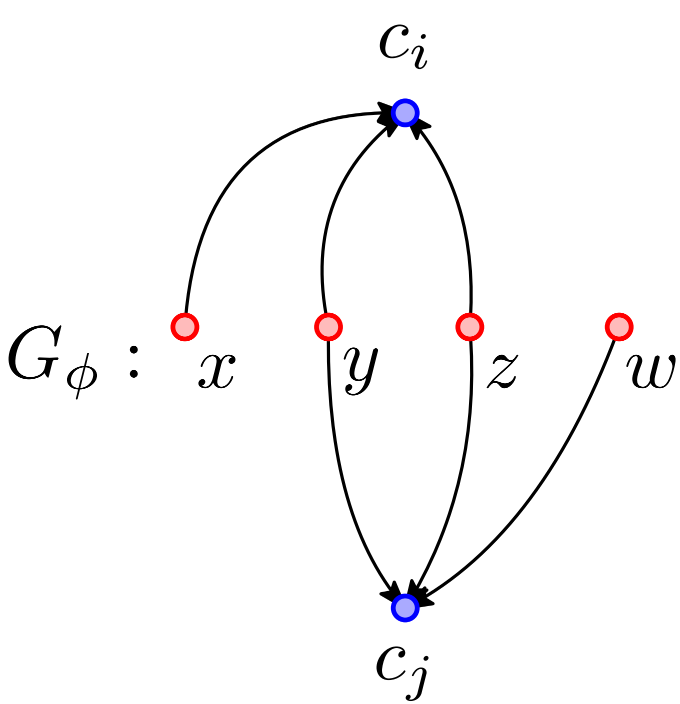


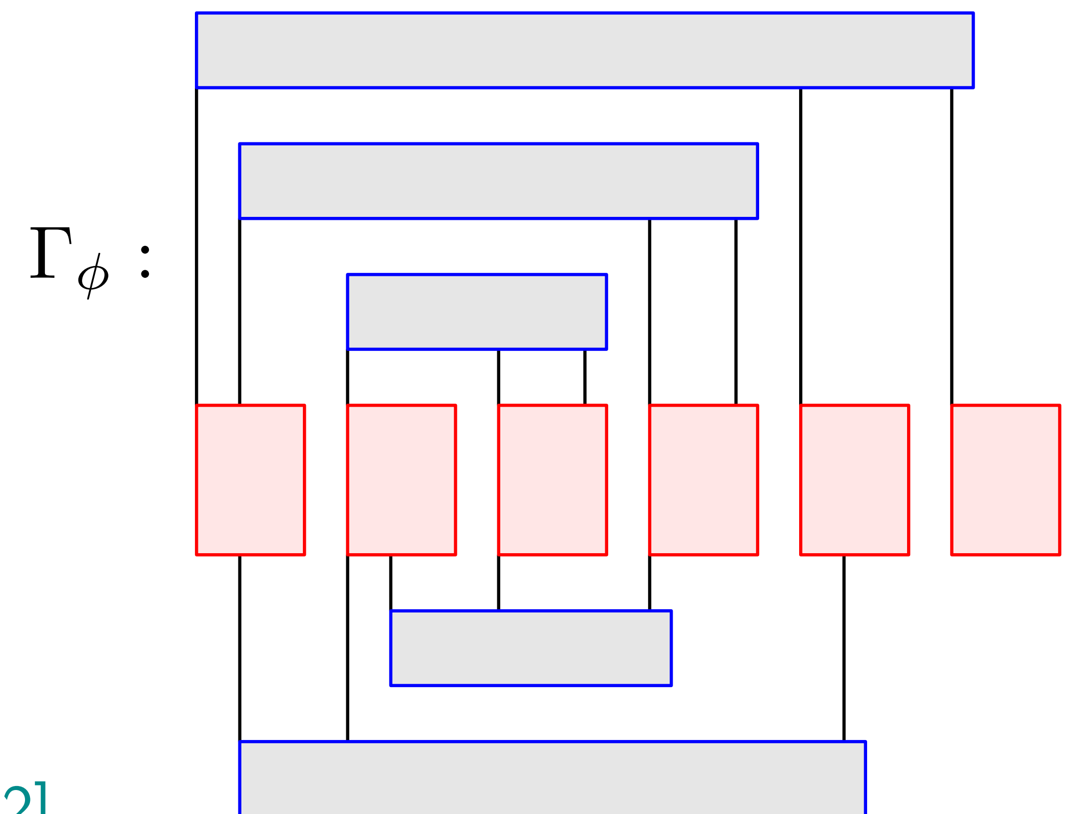
Γ_ϕ :



Planar Monotone 3-SAT

$\phi: \quad \ldots \vee (x \wedge y \wedge z) \vee \ldots \vee (\neg y \wedge \neg z \wedge \neg w) \vee \ldots$





[de Berg & Abel, 2012]

The problem is NP-complete, even with a Monotone Rectilinear Representation

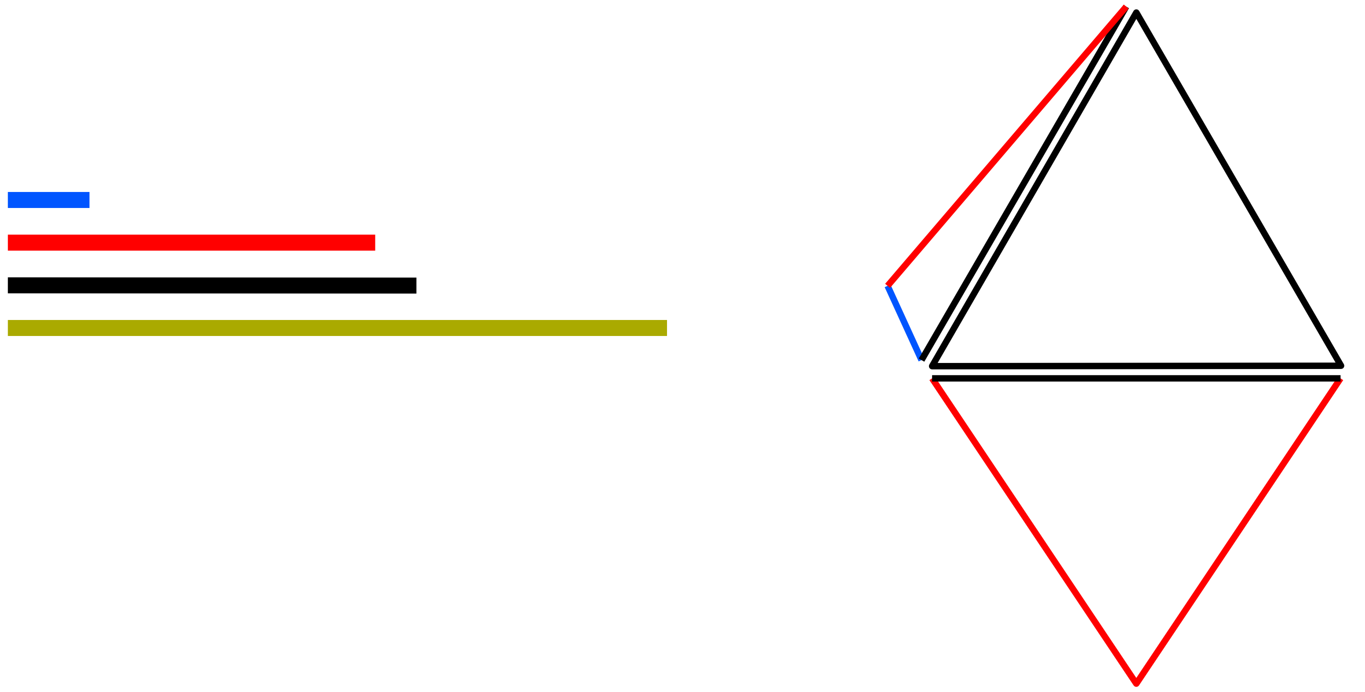
The four distances

The four distances

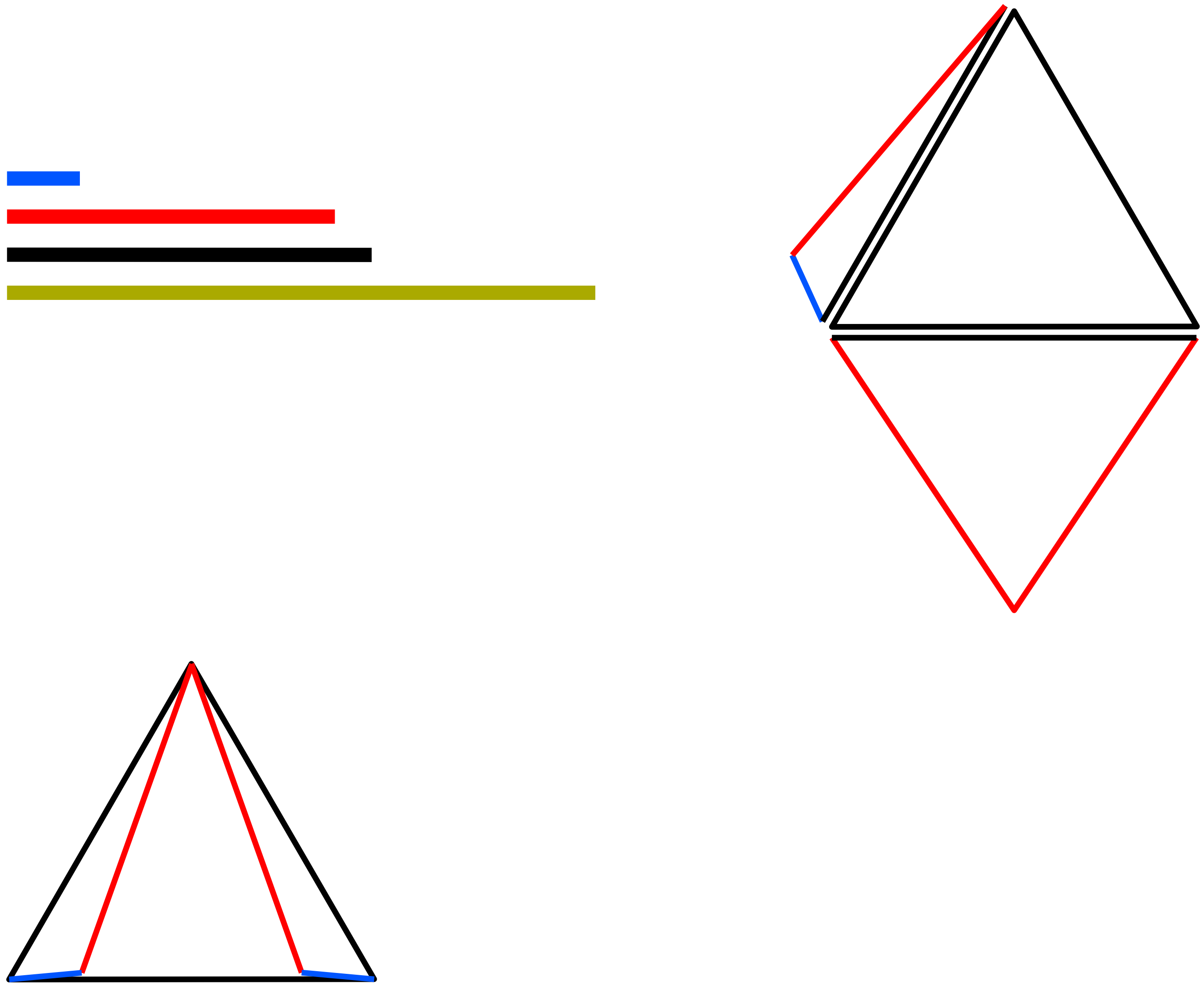
The four distances

The four distances

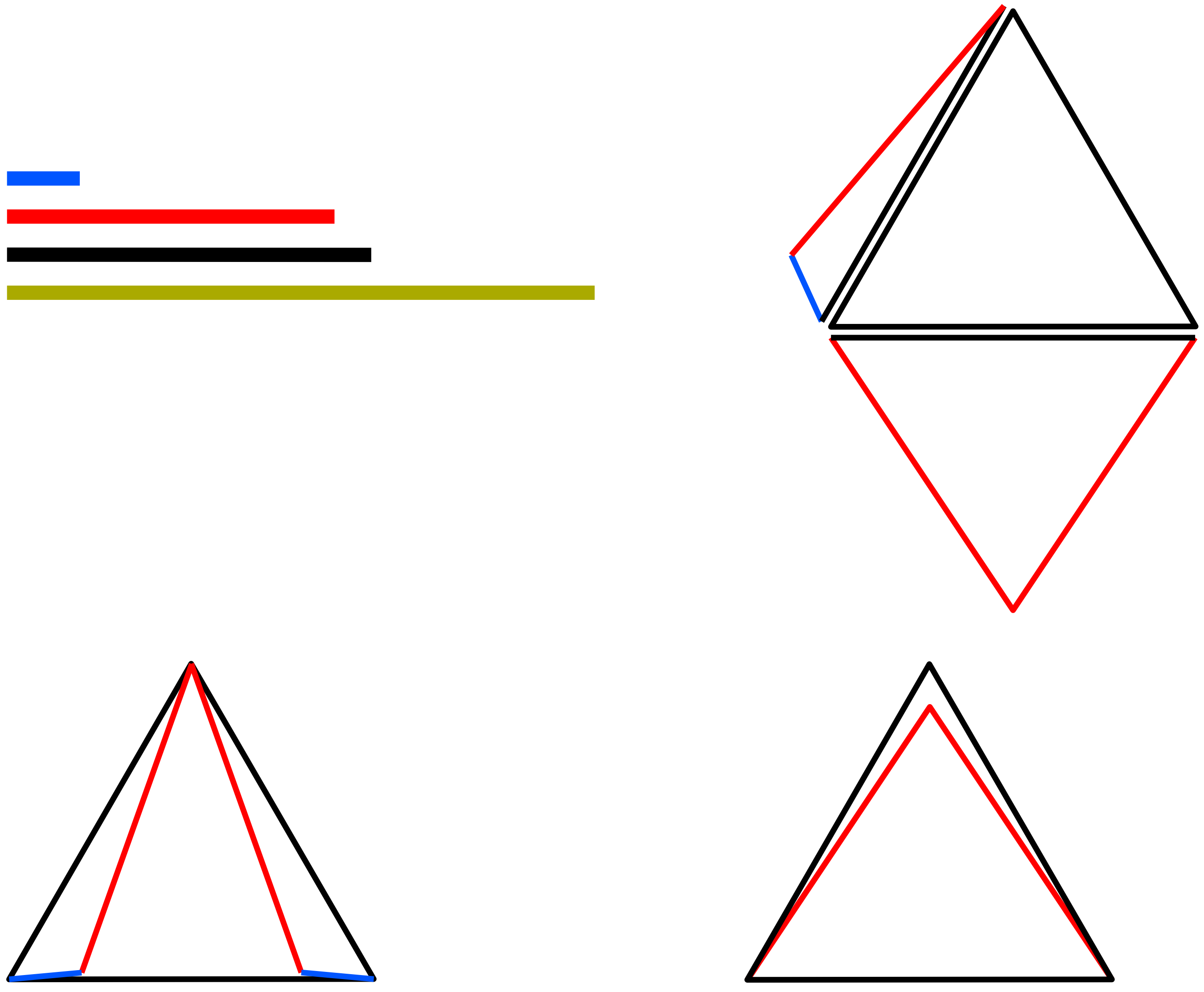
The four distances



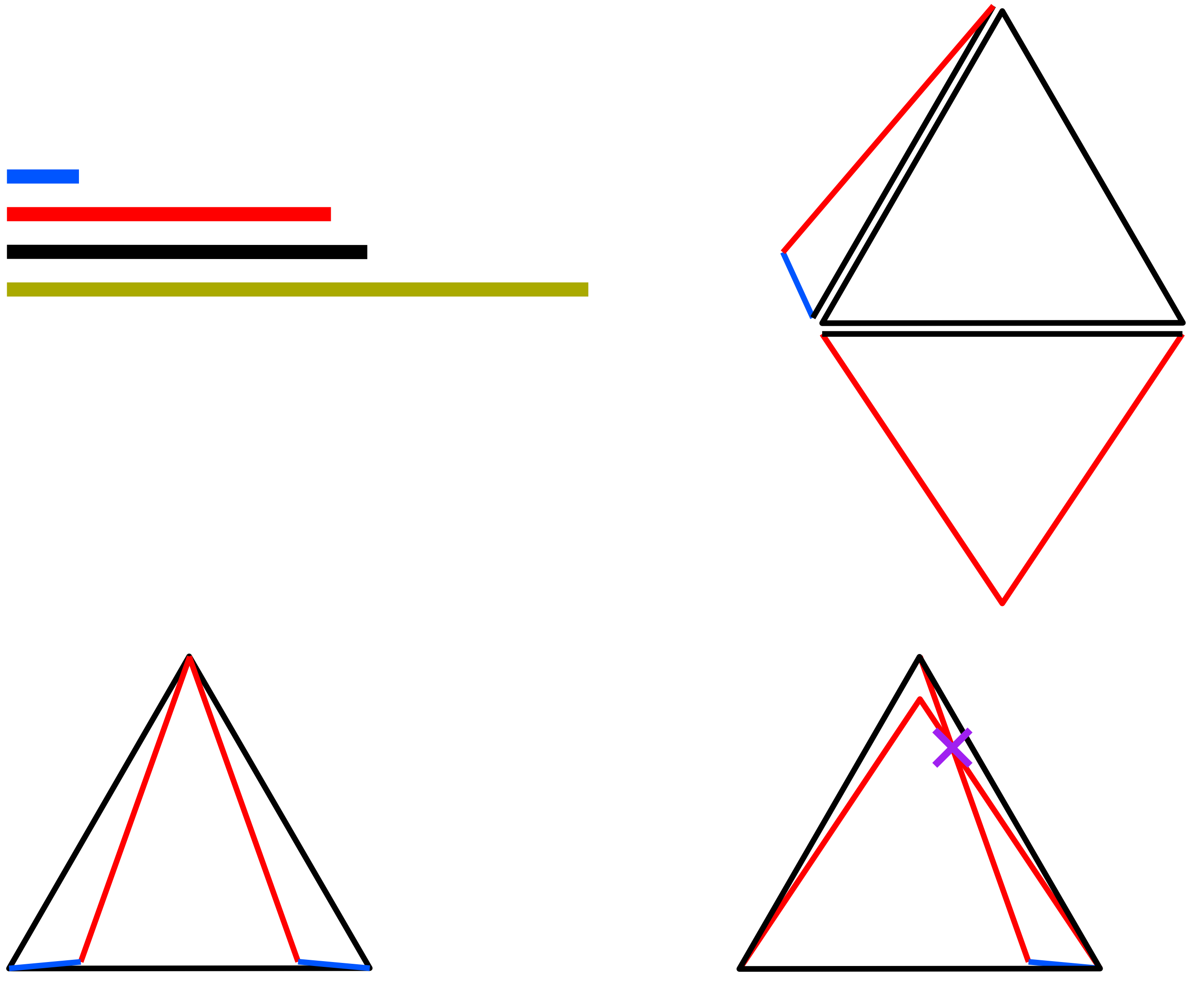
The four distances



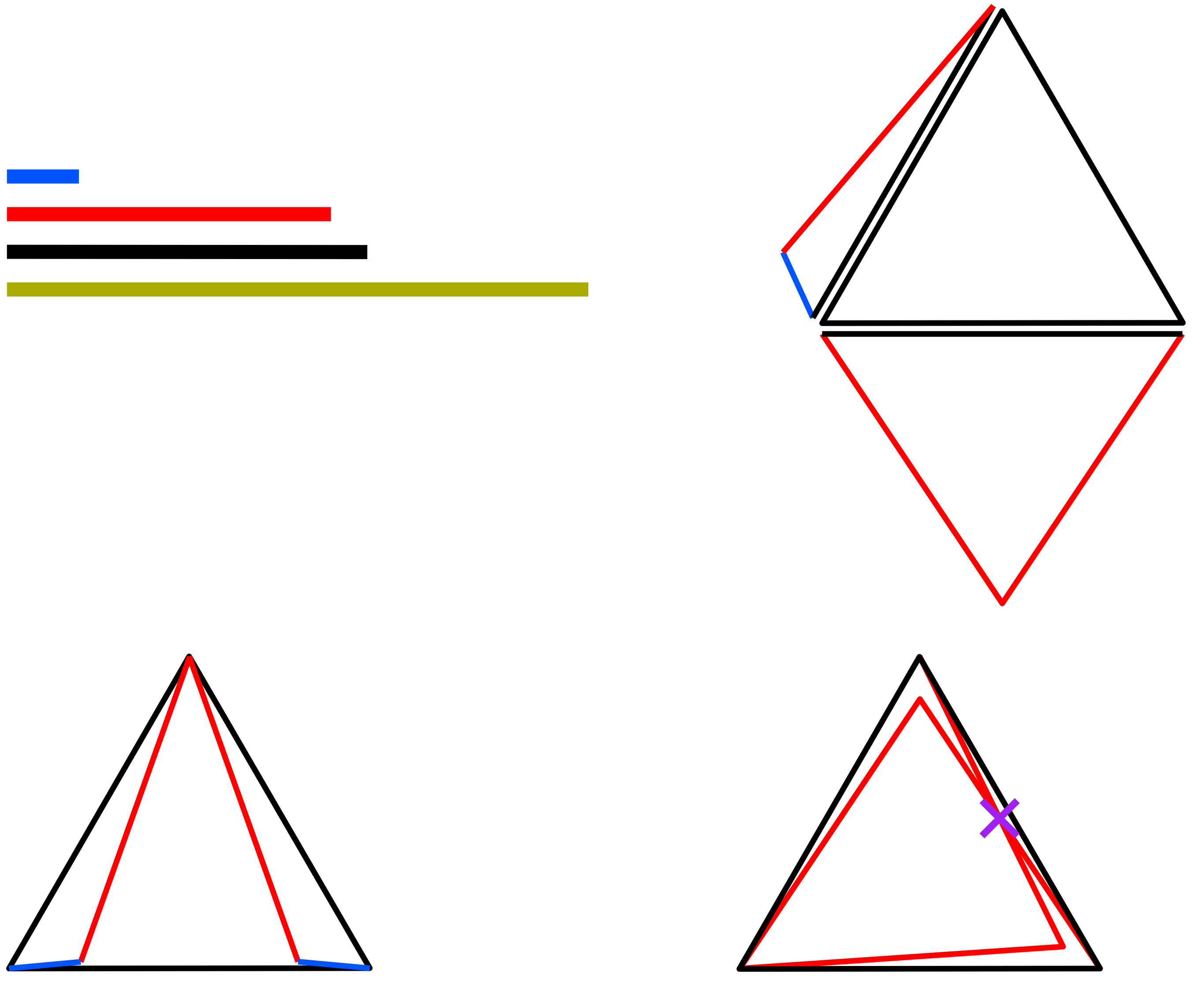
The four distances



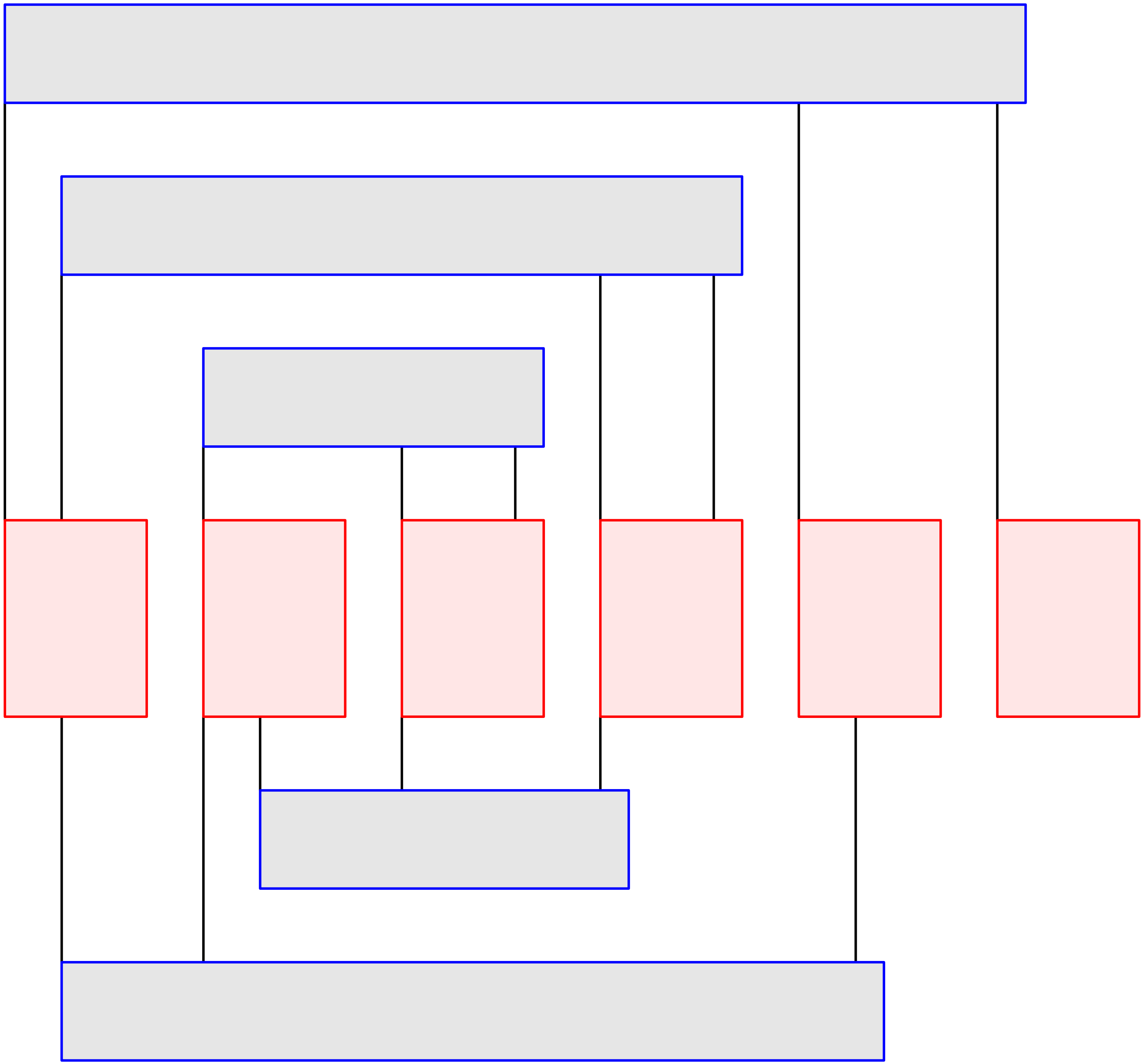
The four distances



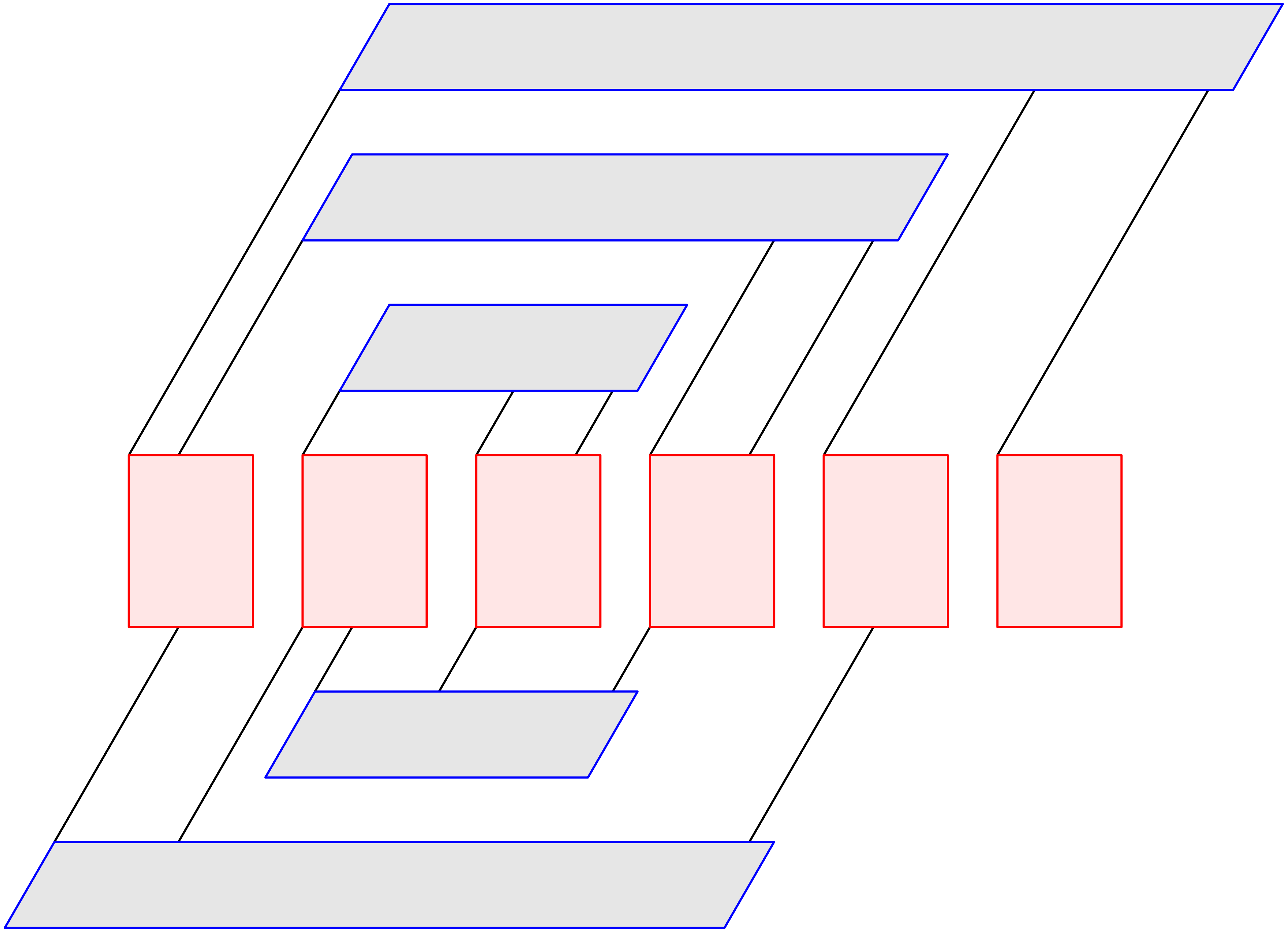
The four distances



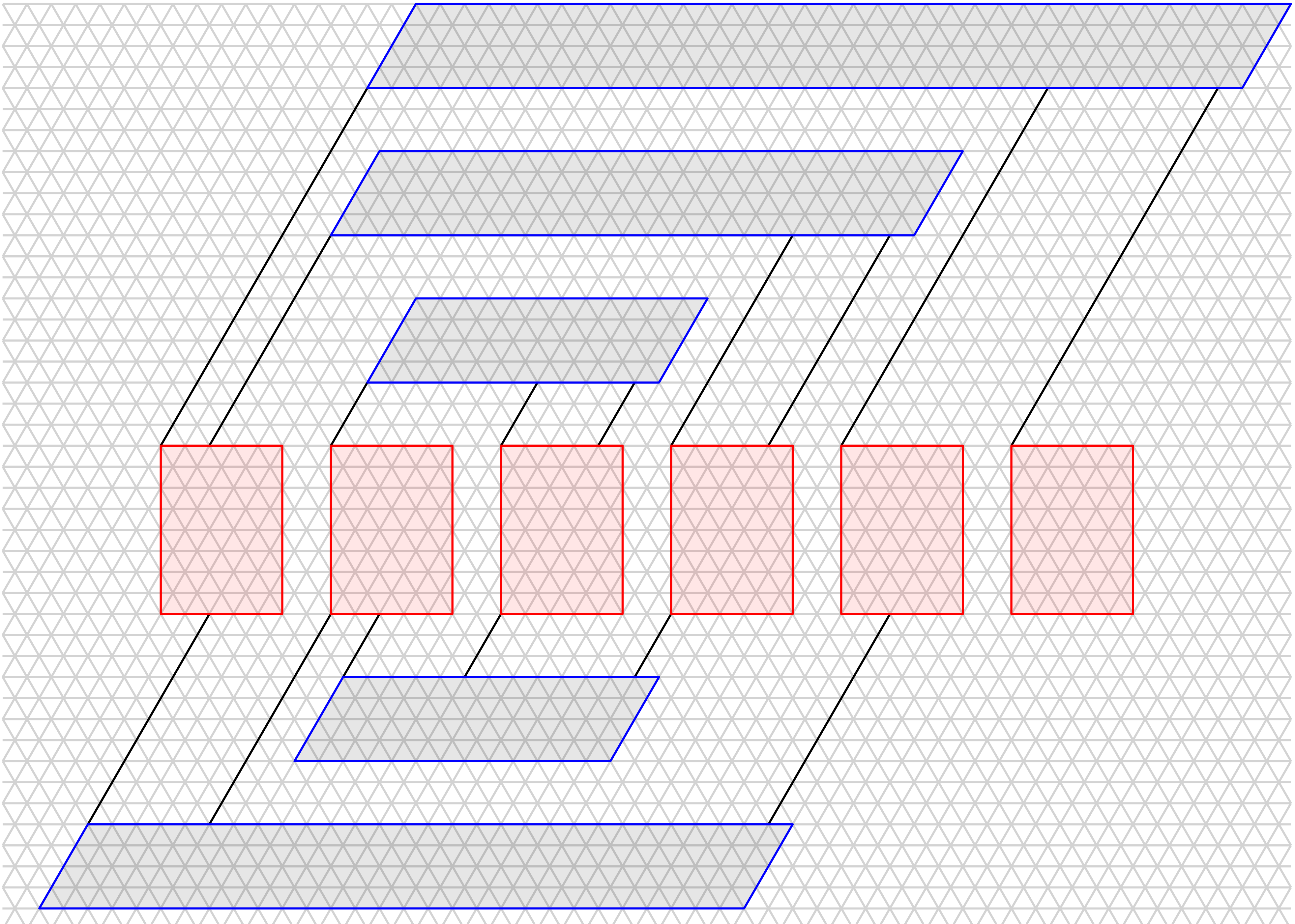
The gadgets



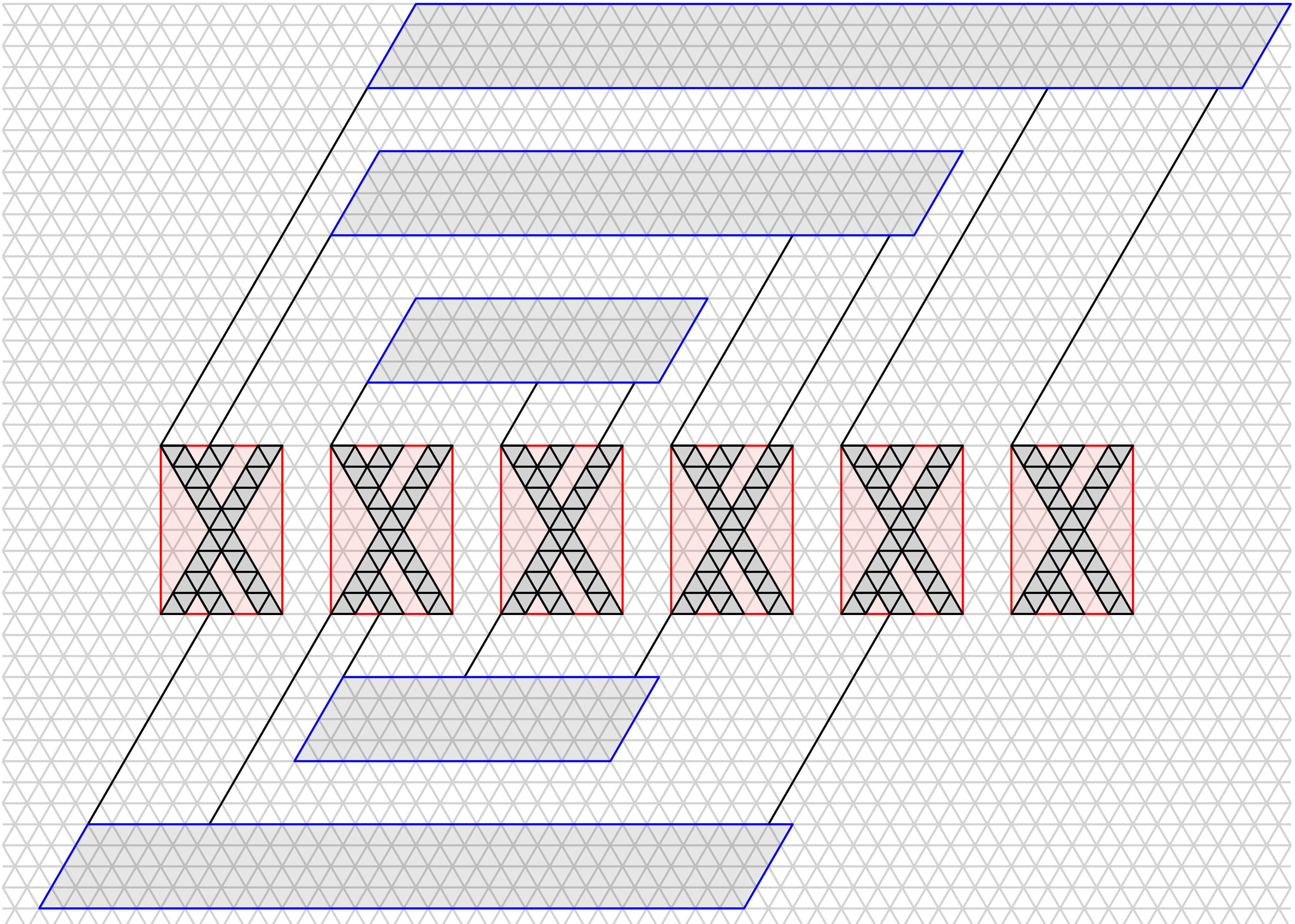
The gadgets



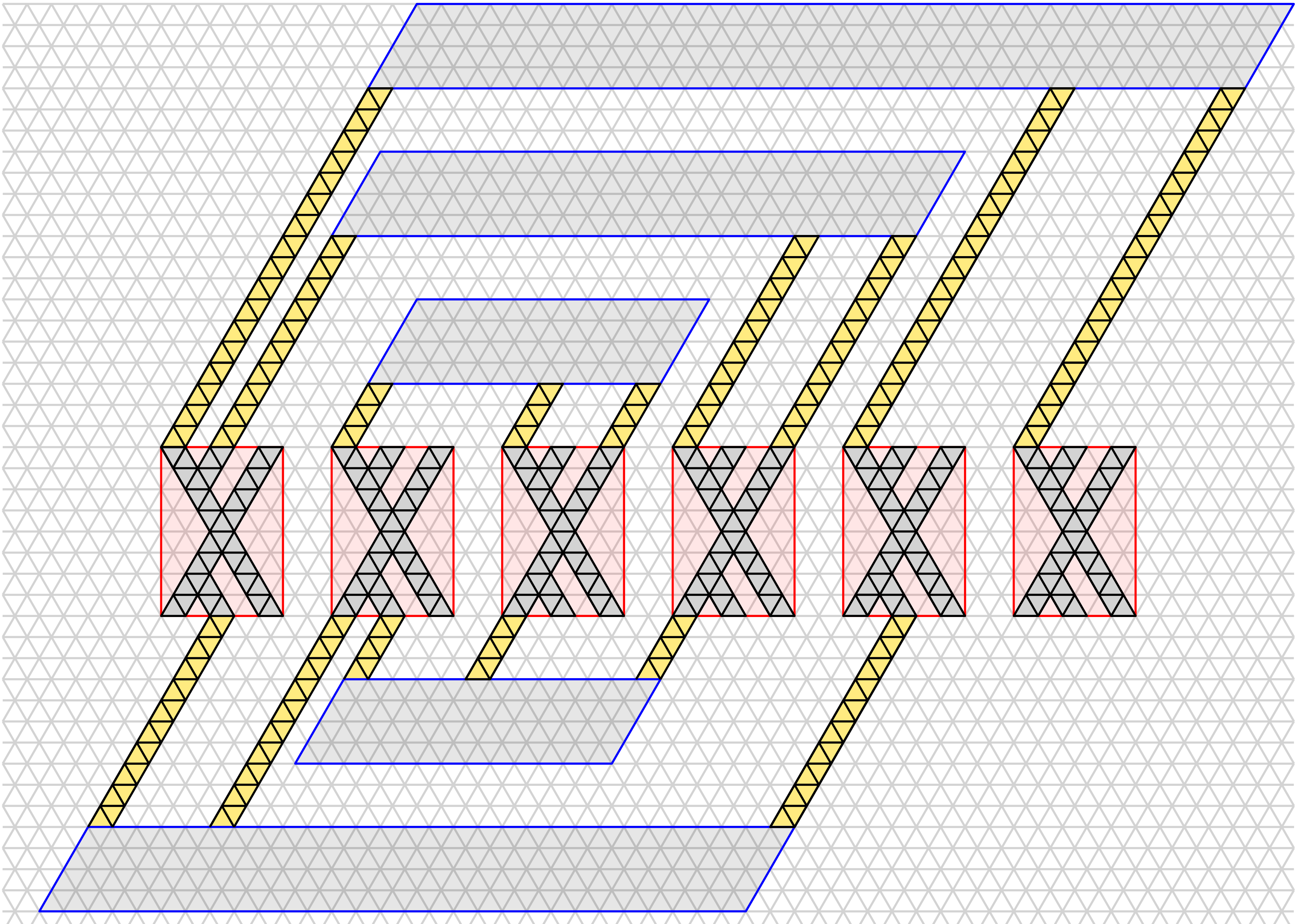
The gadgets



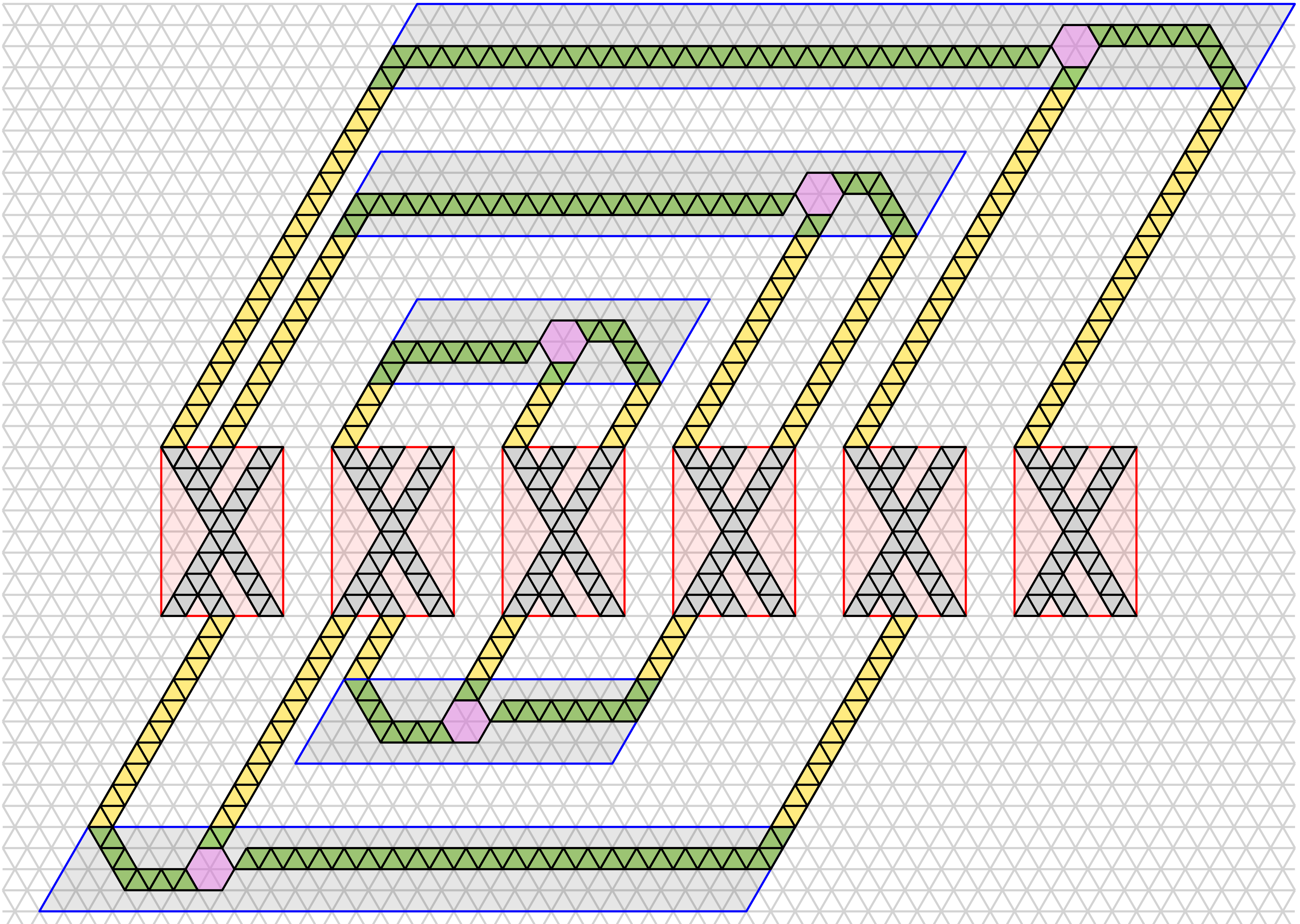
The gadgets



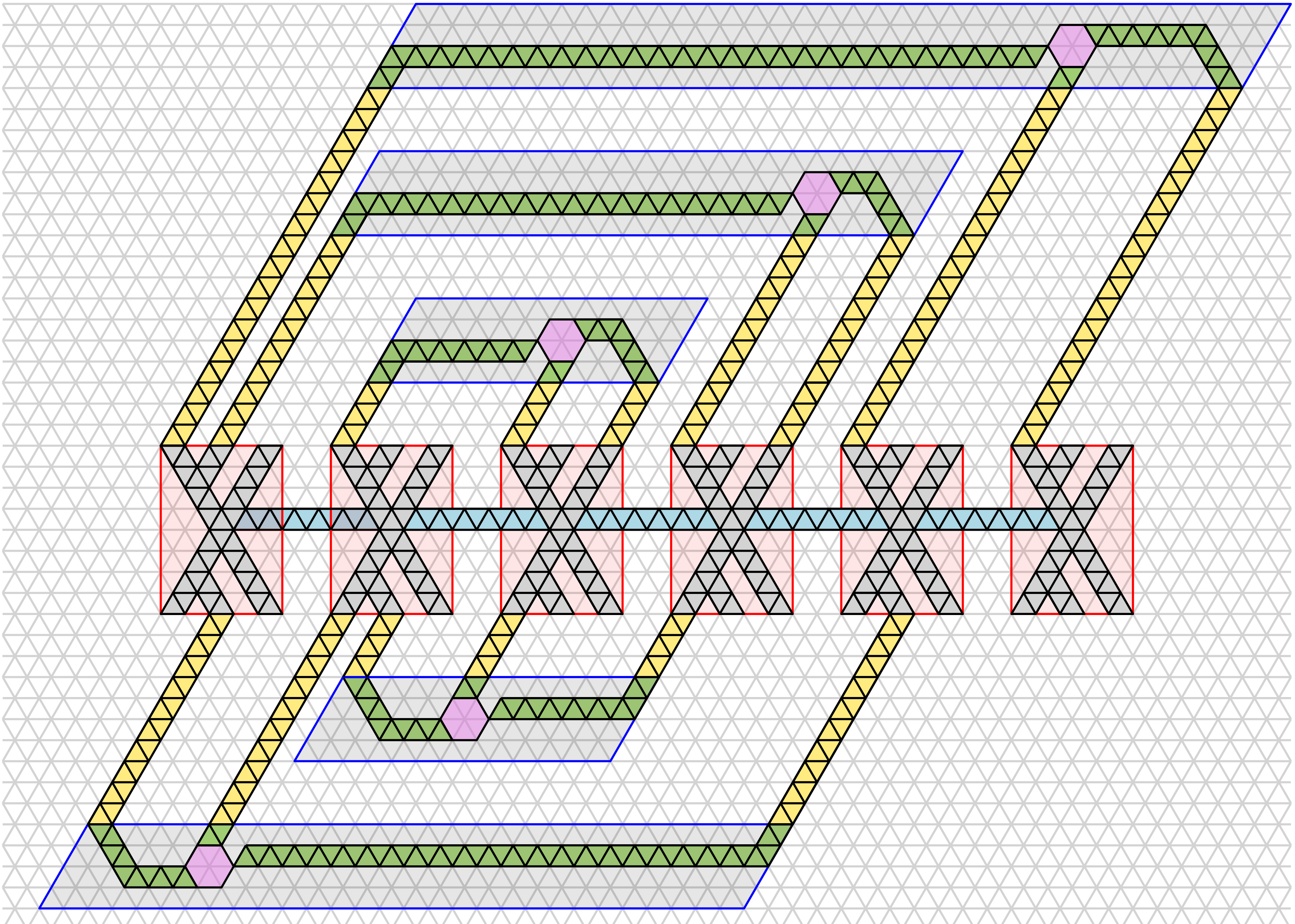
The gadgets



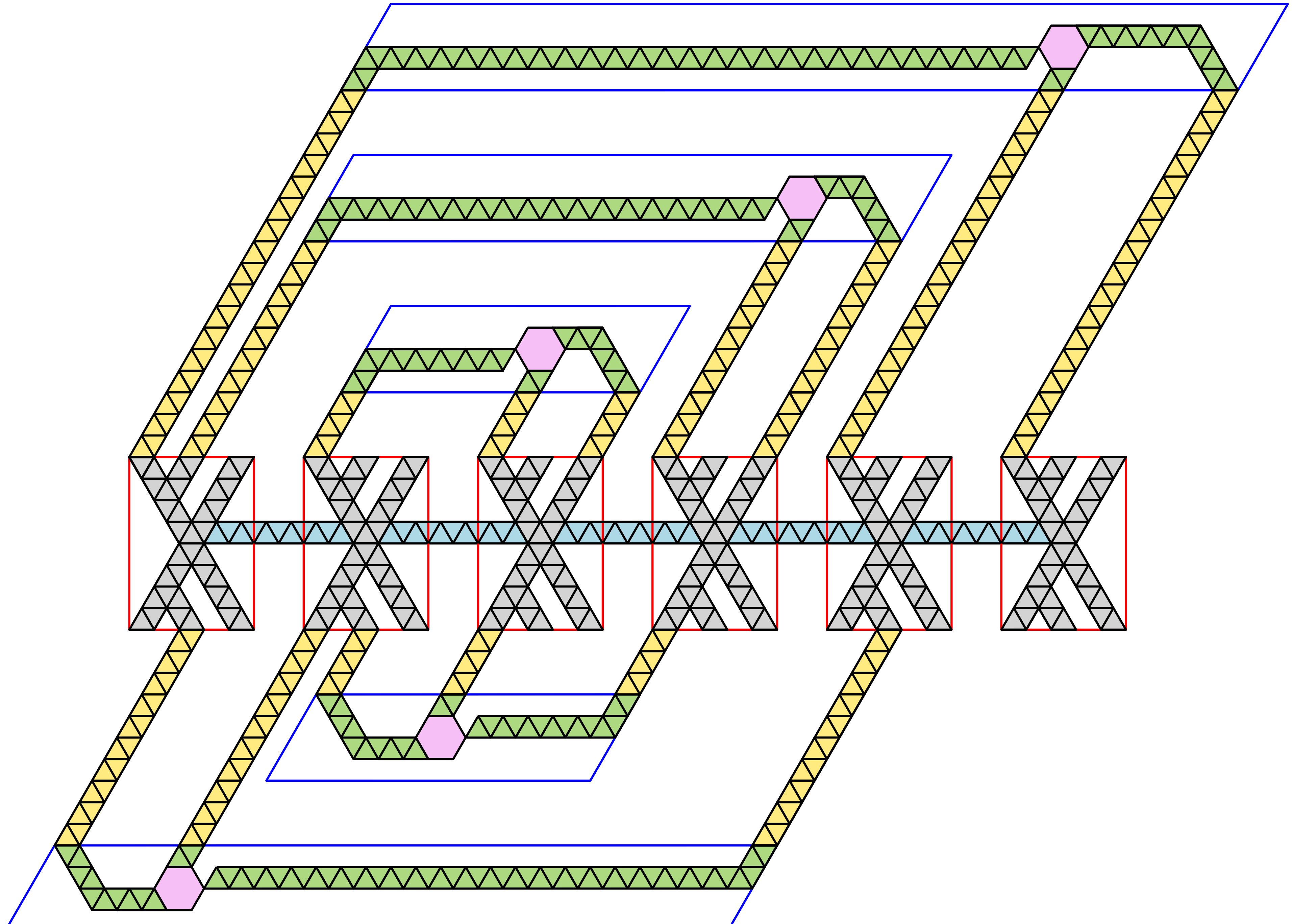
The gadgets



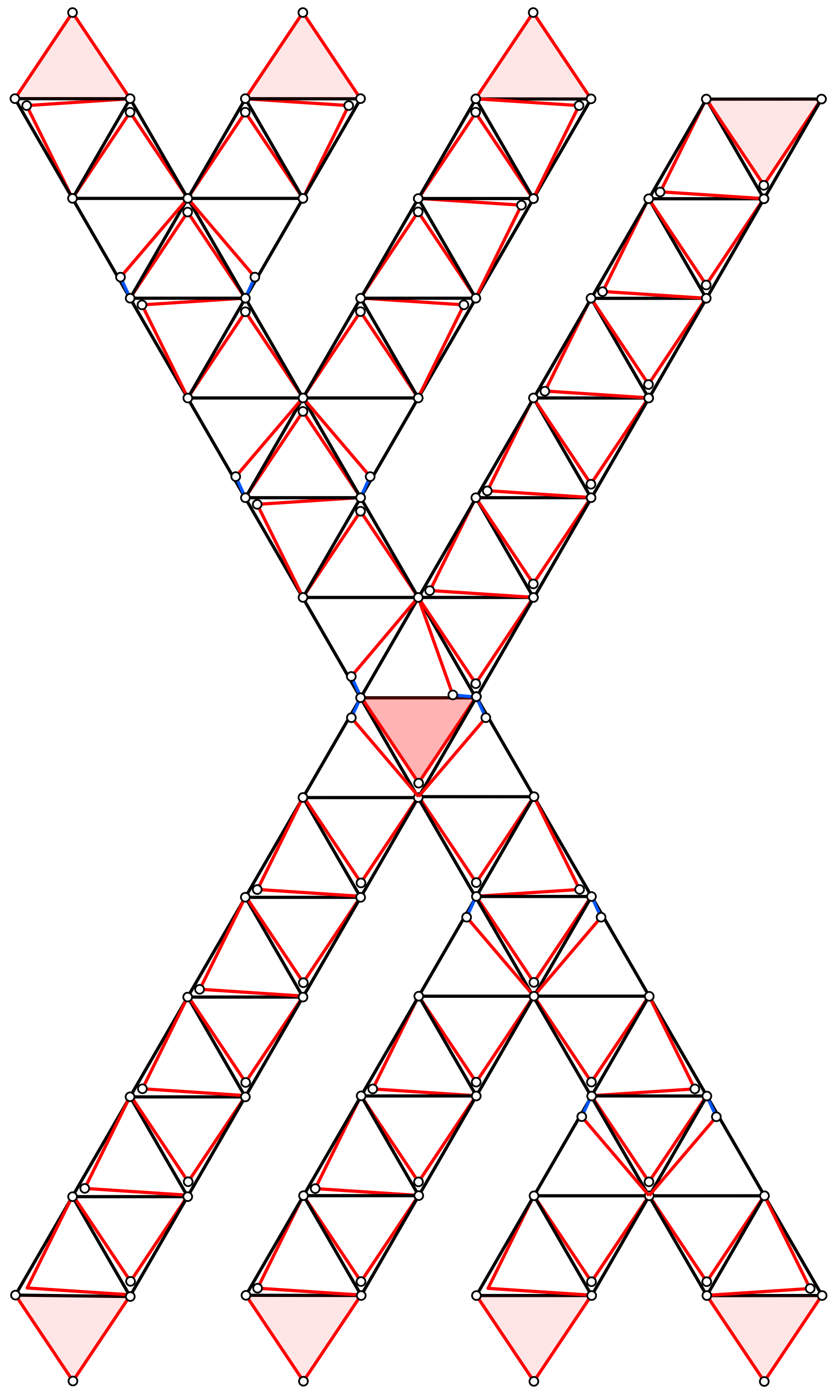
The gadgets



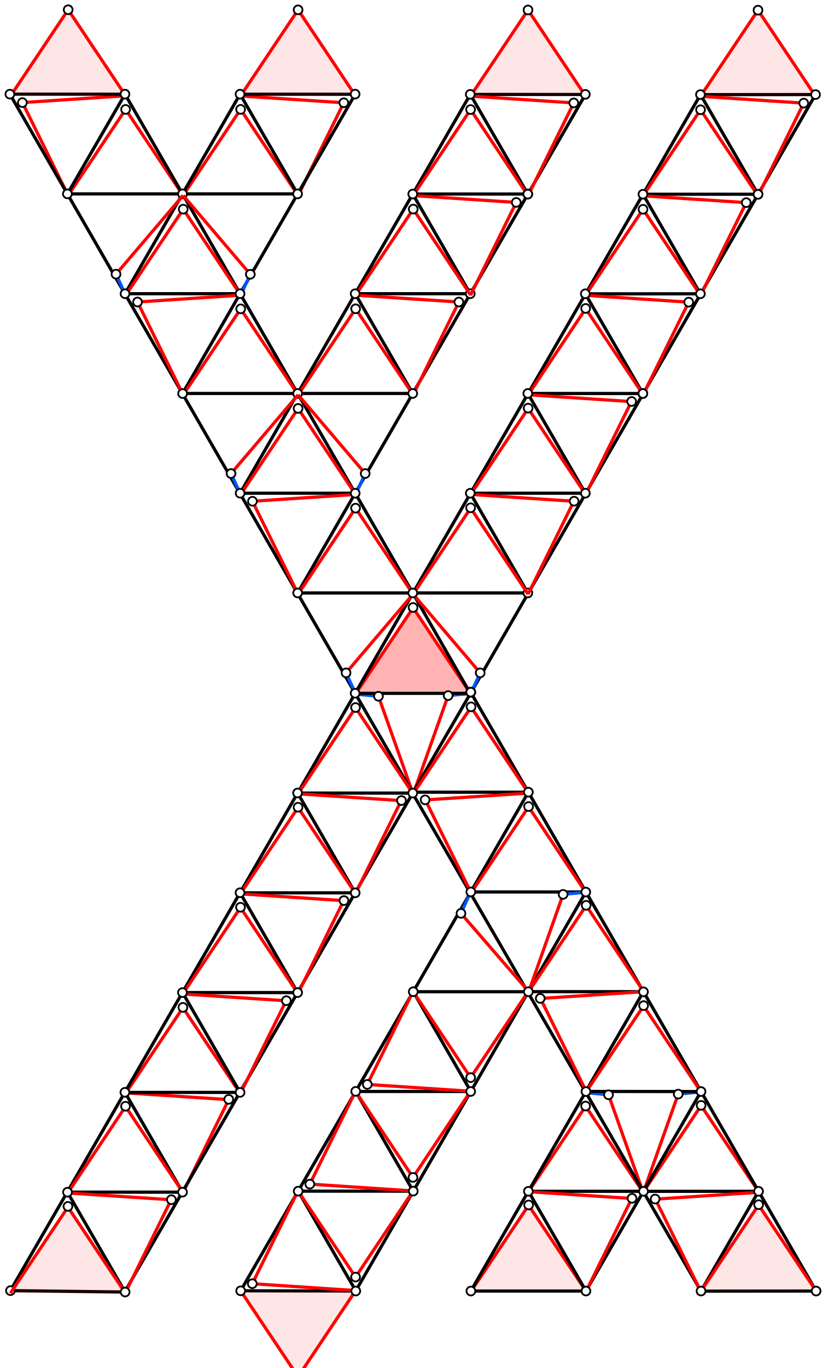
The gadgets



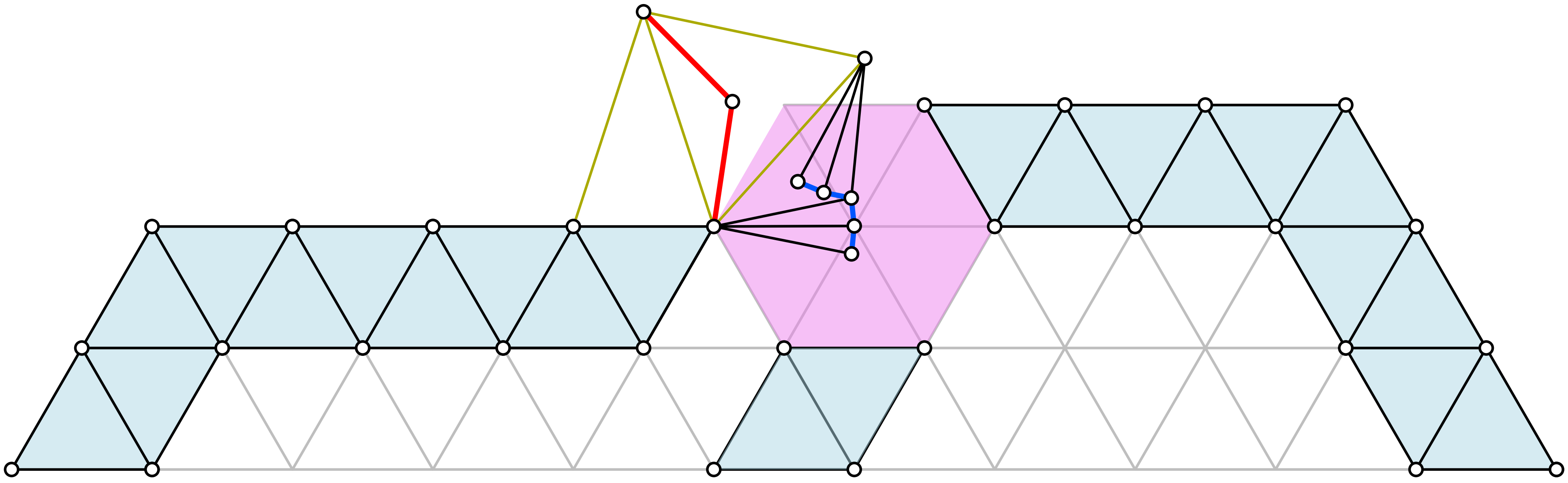
Variable gadget



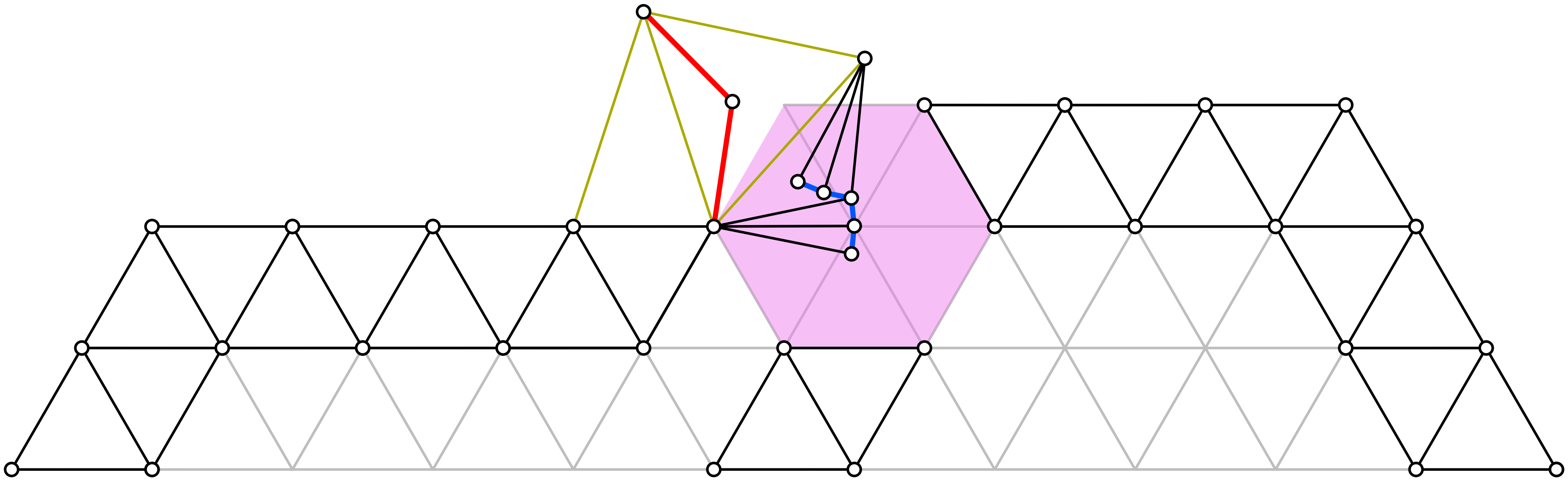
Variable gadget



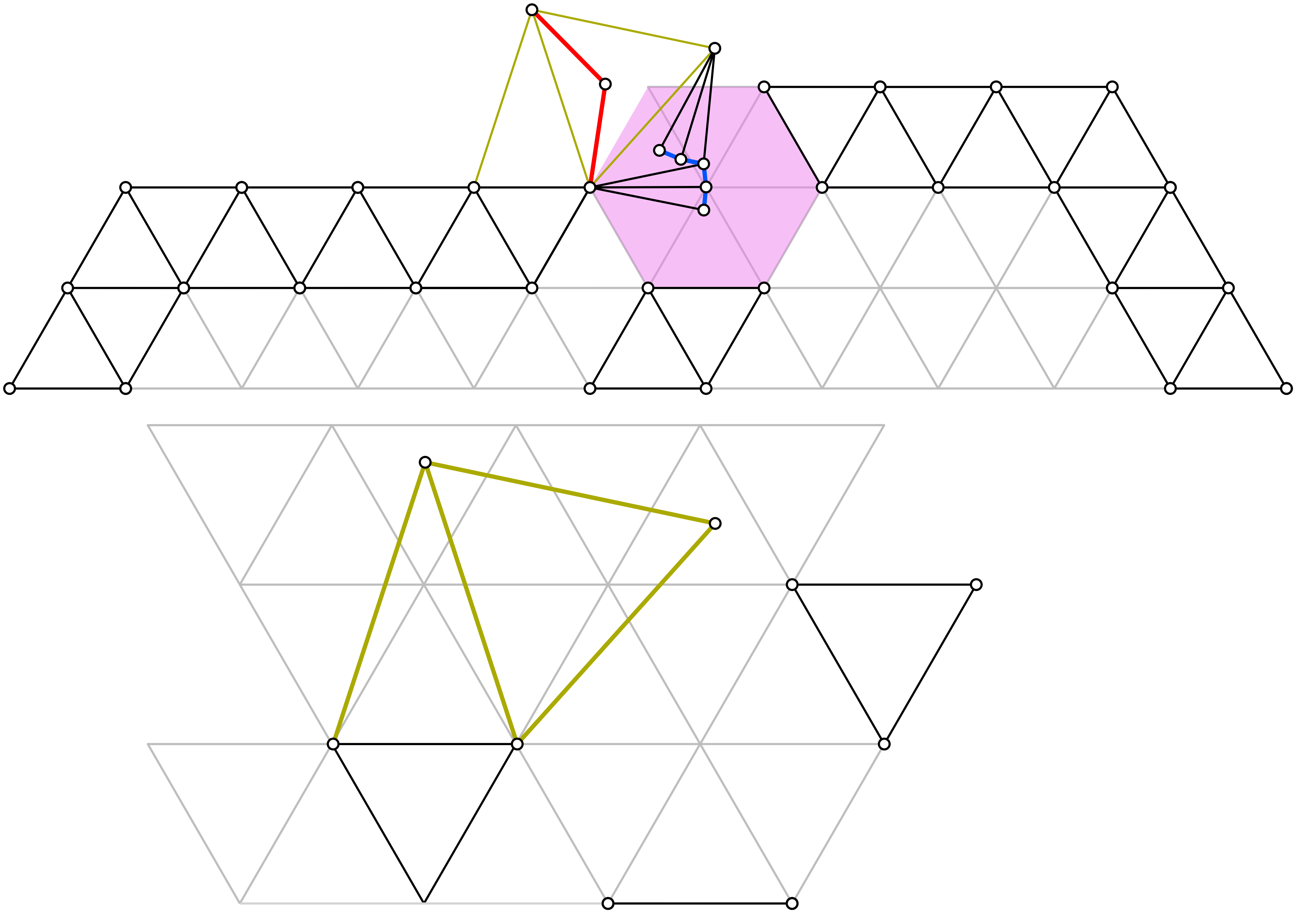
Clause gadget



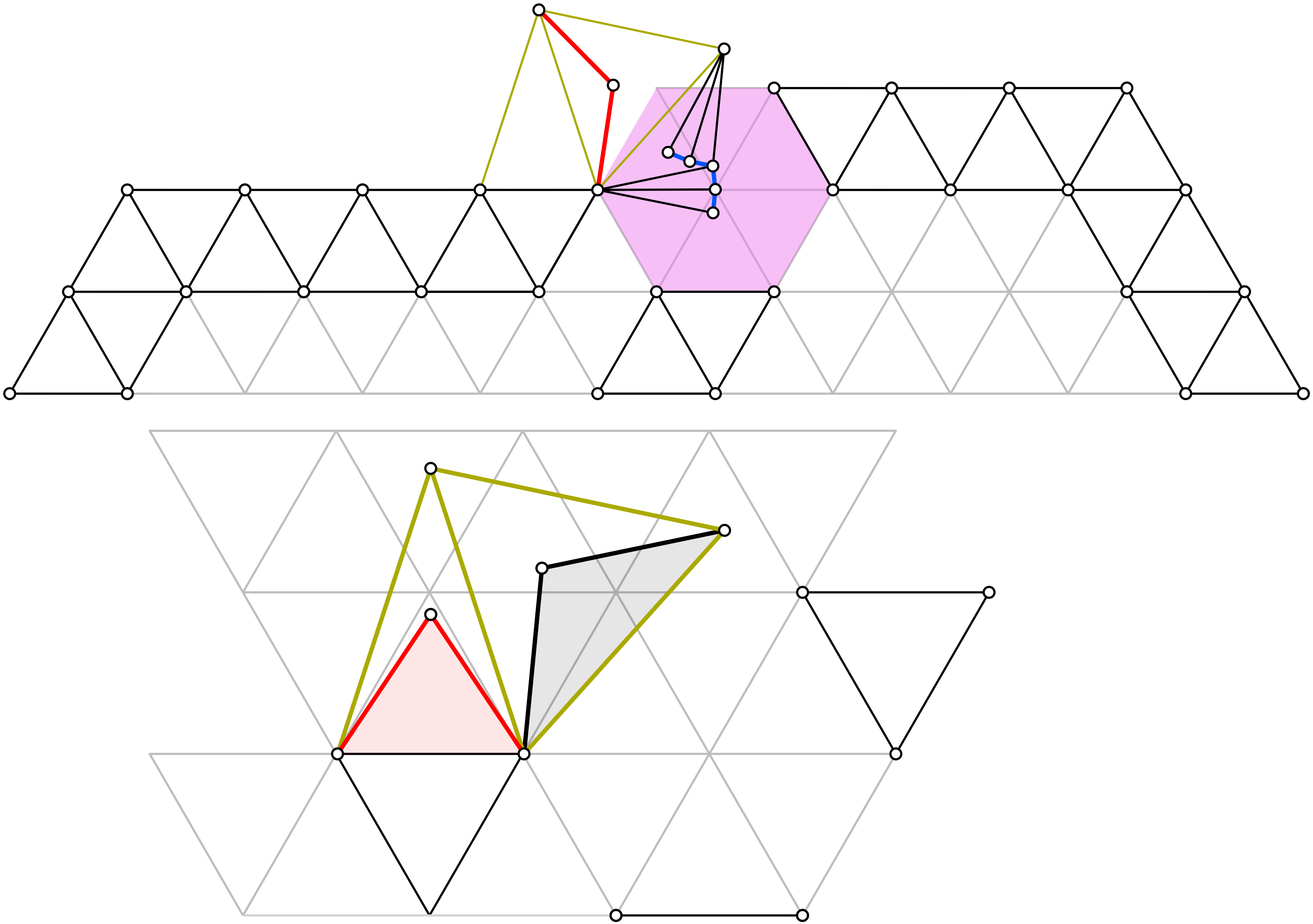
Clause gadget



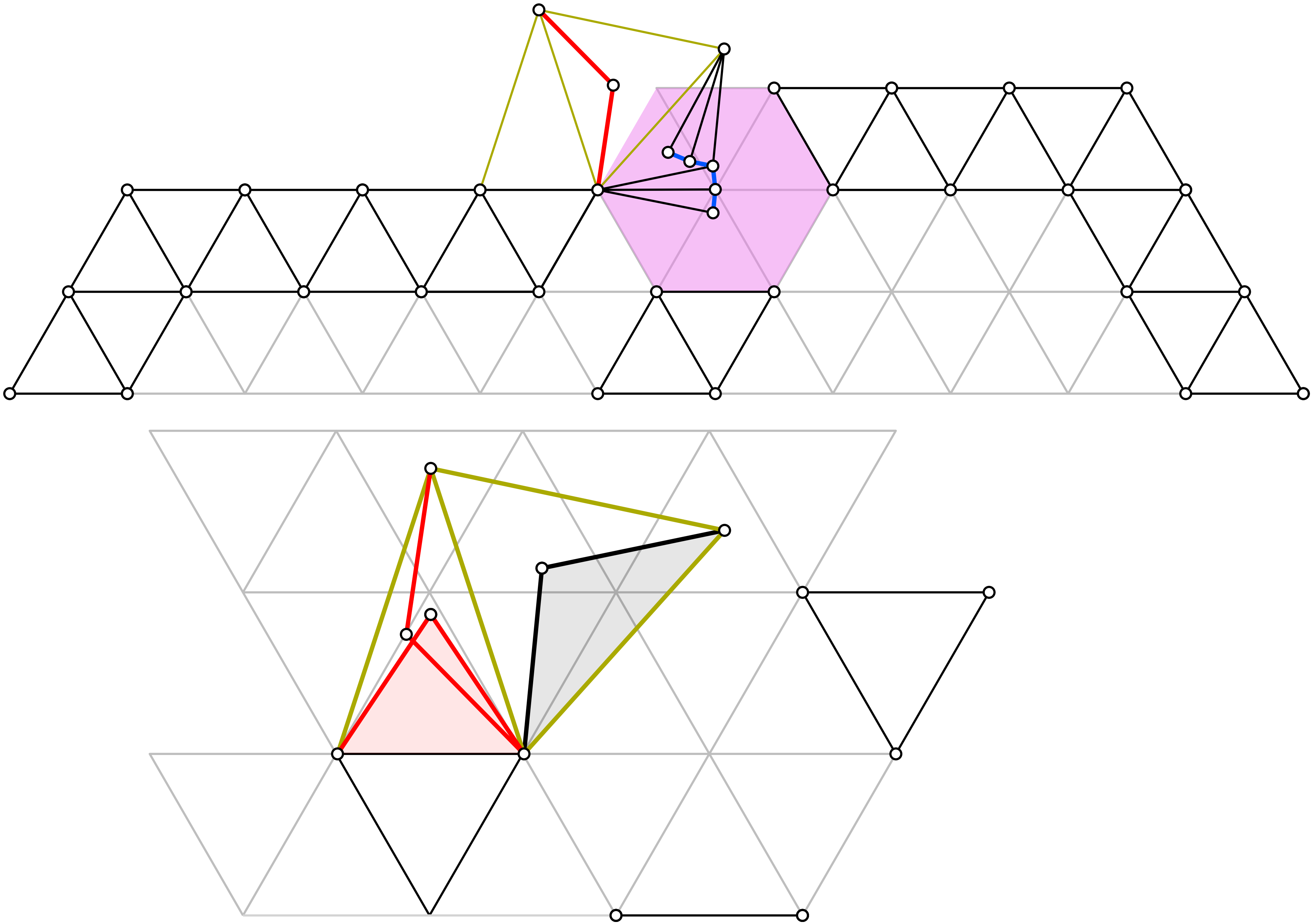
Clause gadget



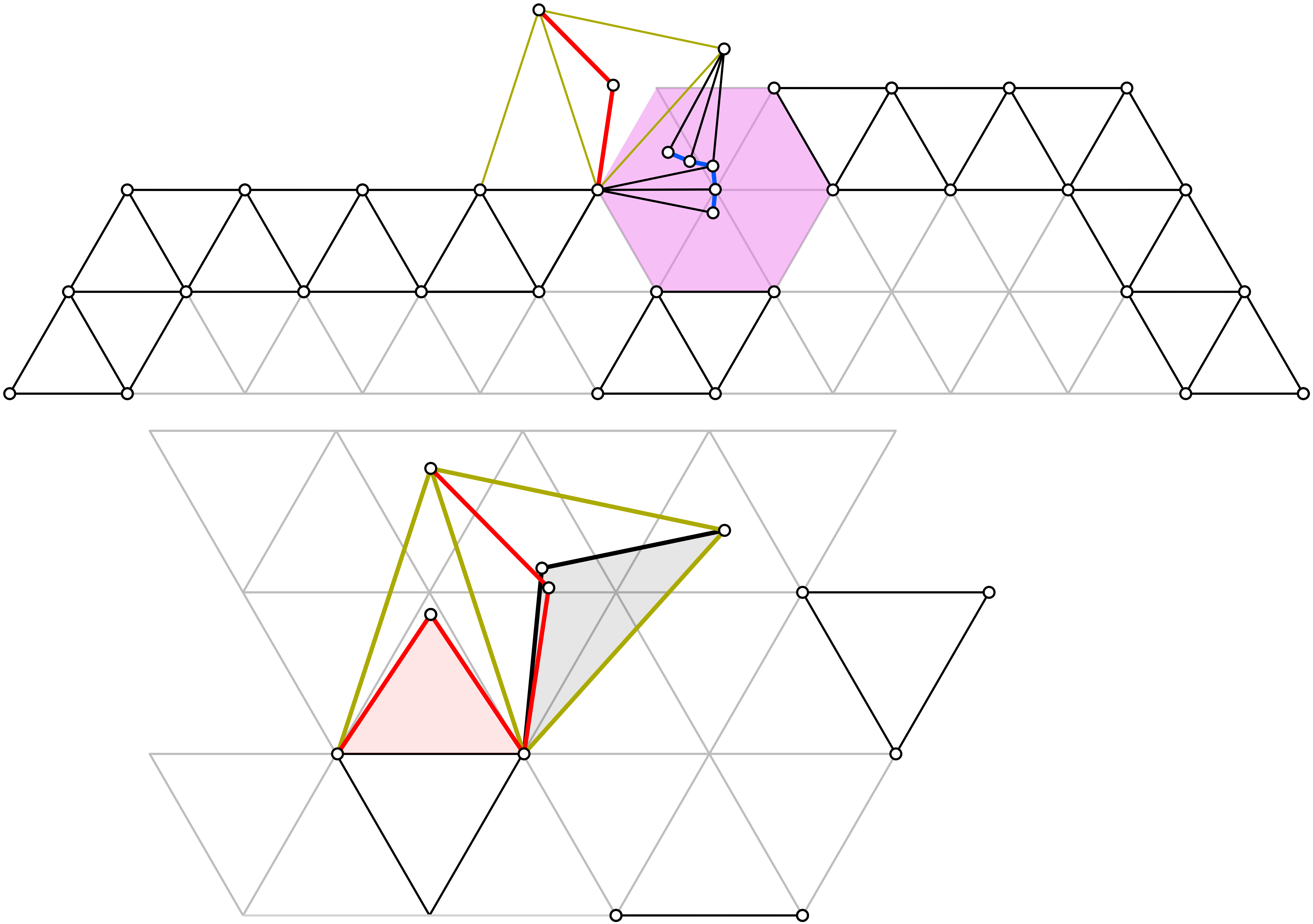
Clause gadget



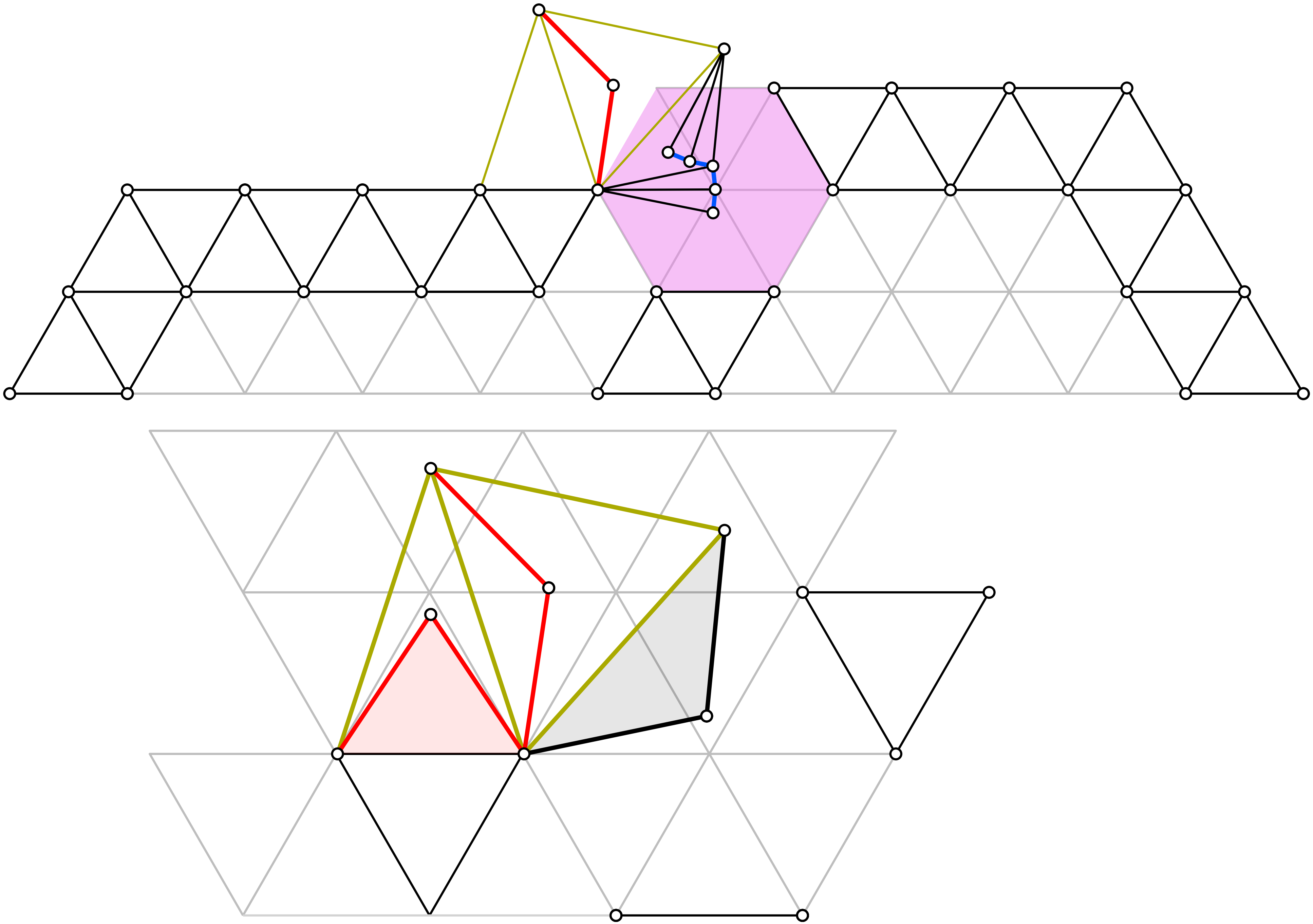
Clause gadget



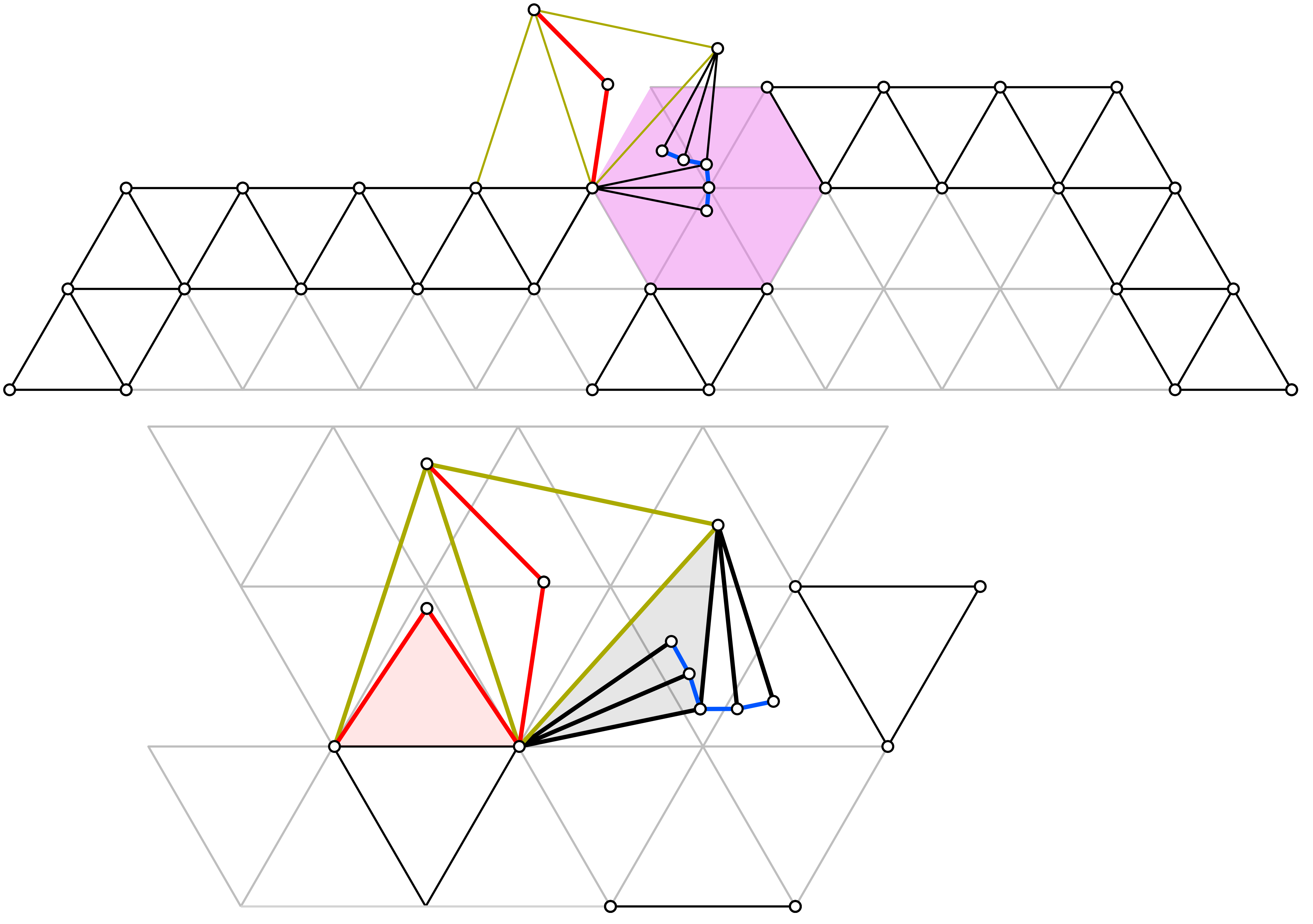
Clause gadget



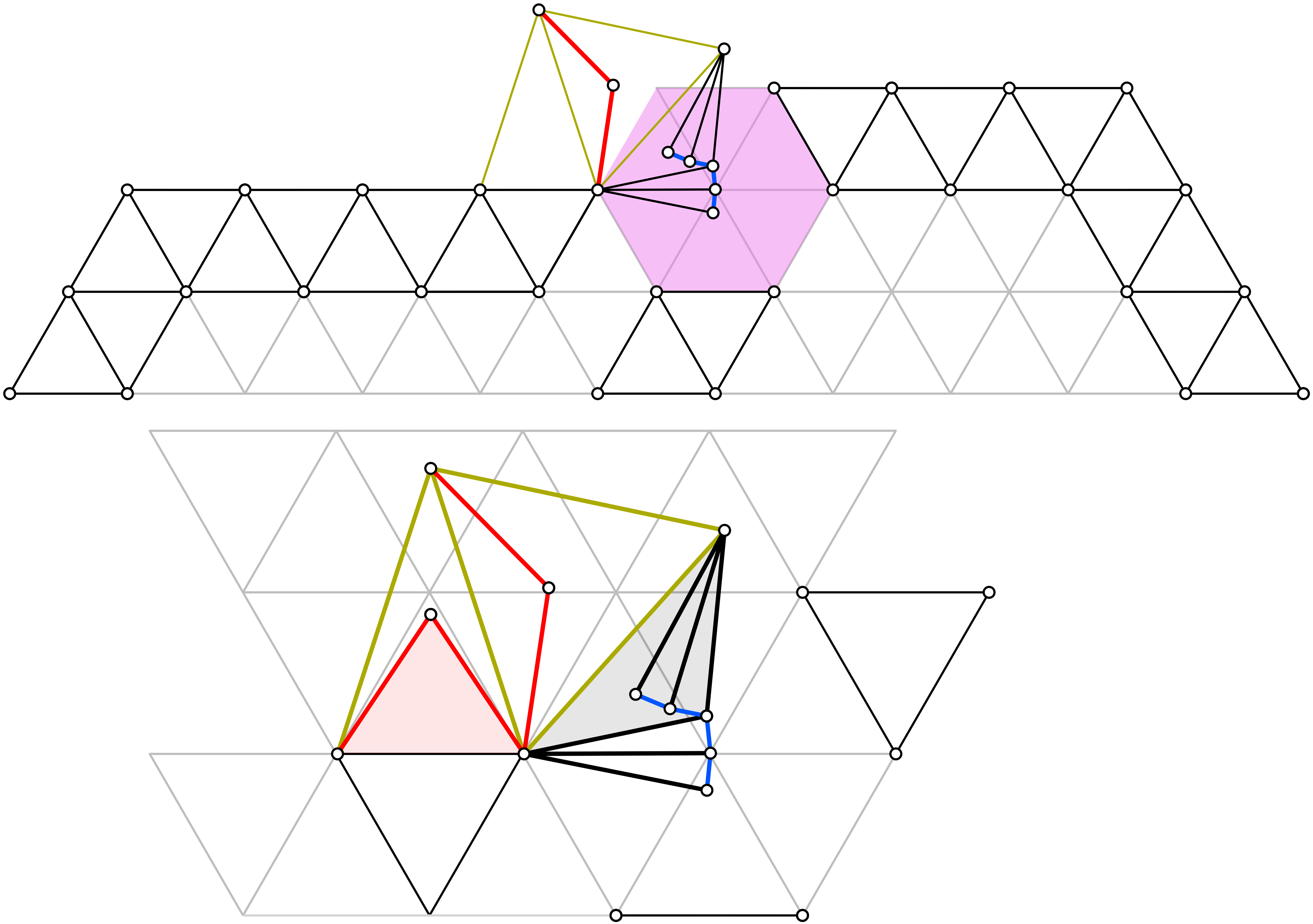
Clause gadget



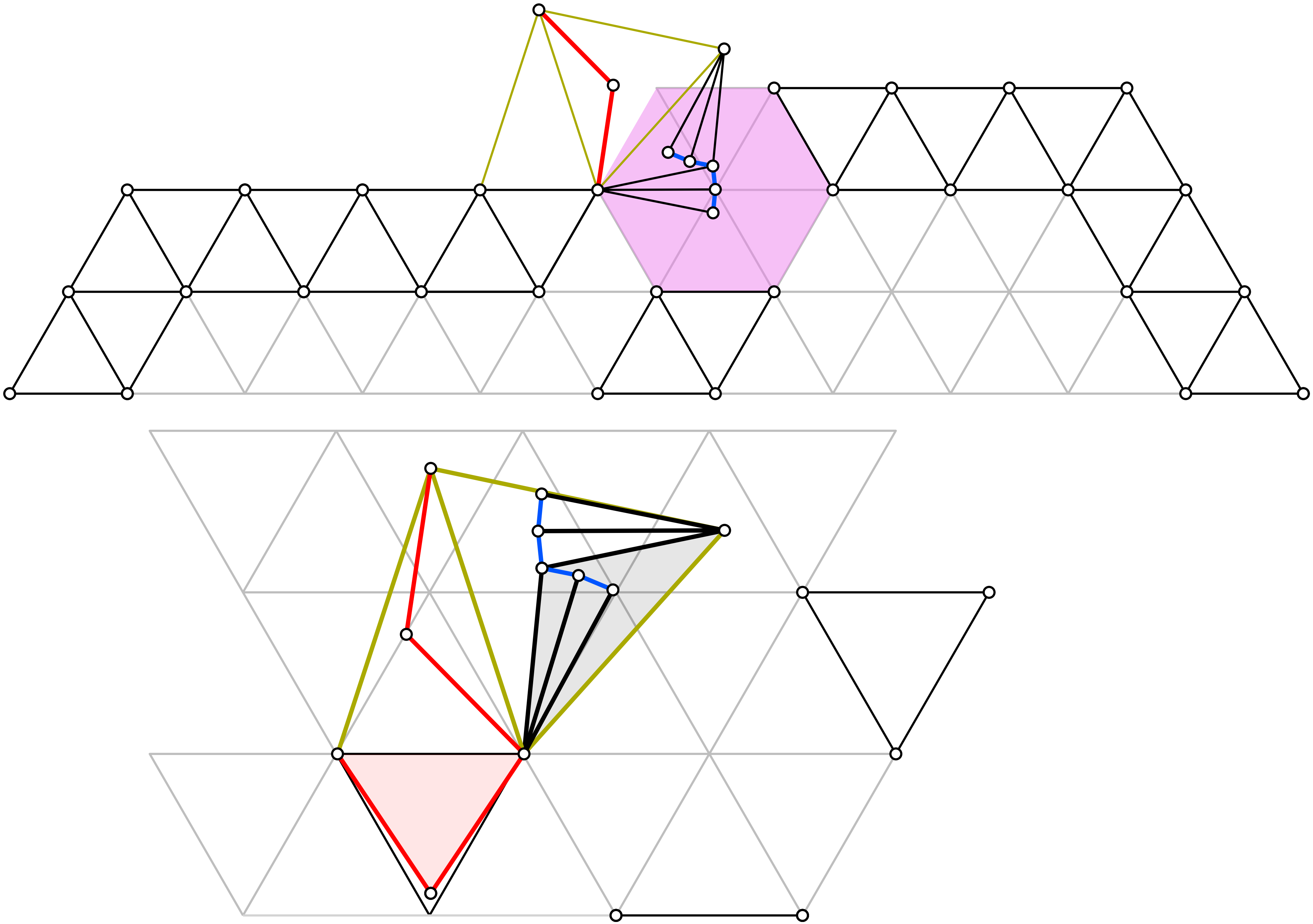
Clause gadget



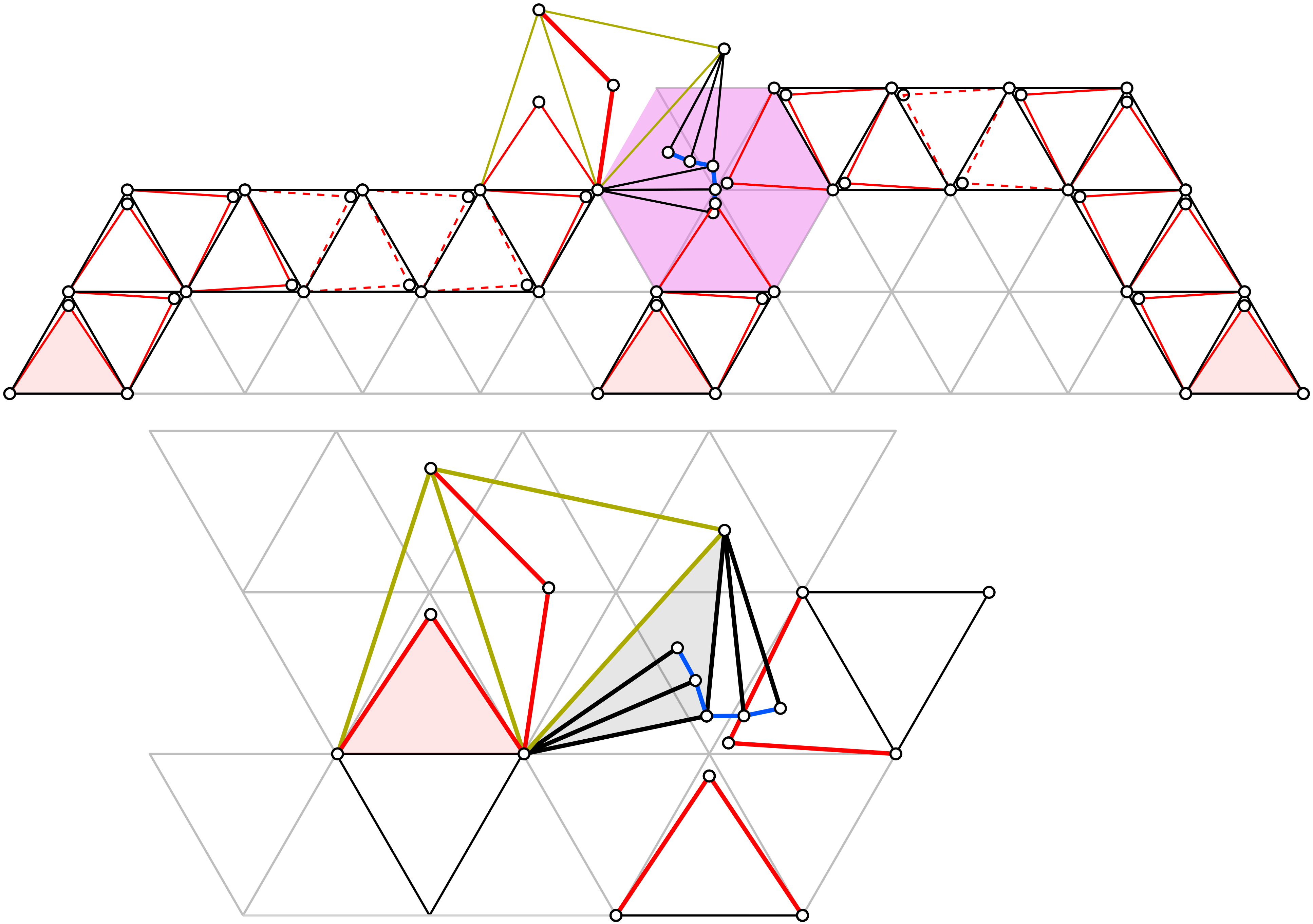
Clause gadget



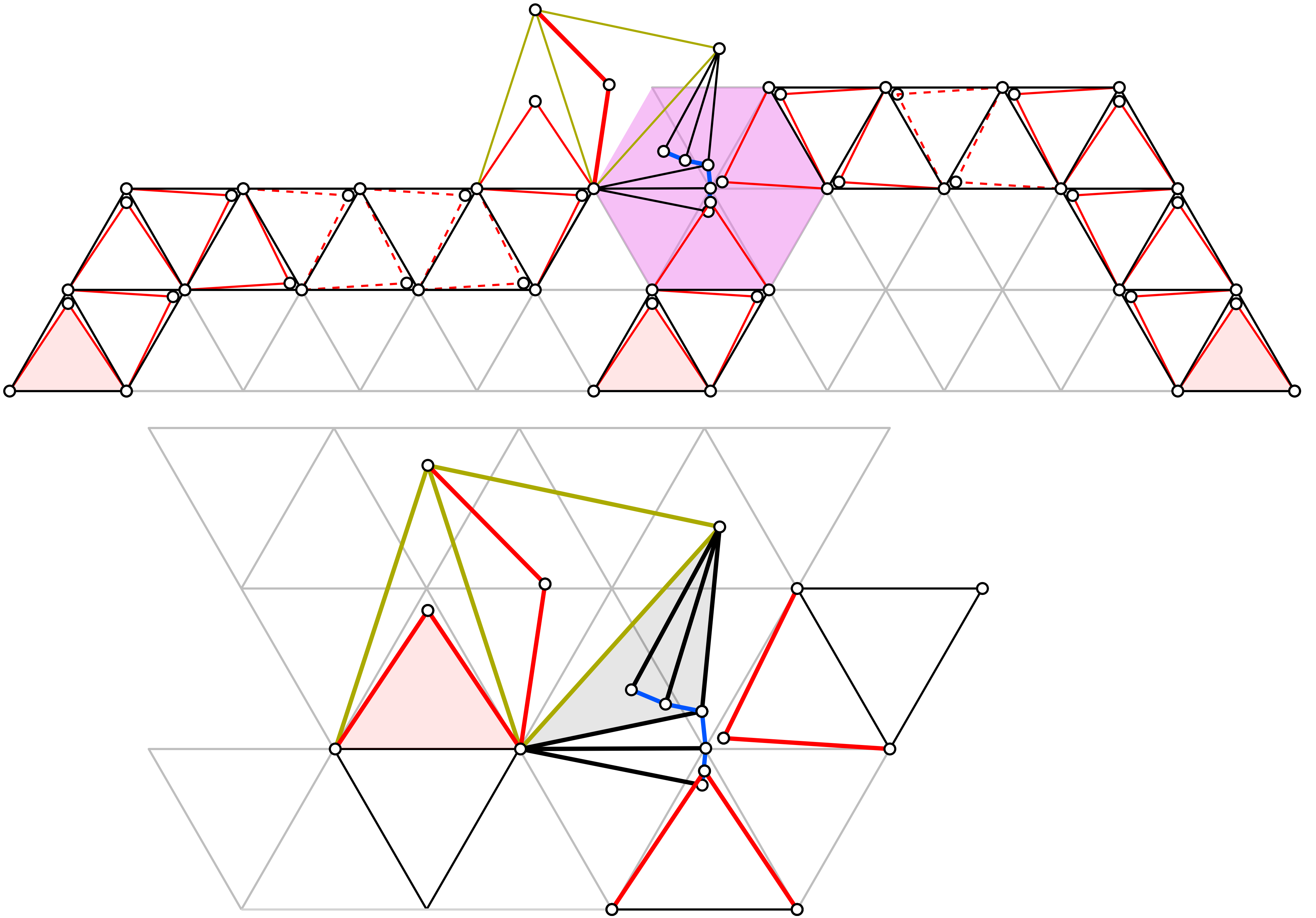
Clause gadget



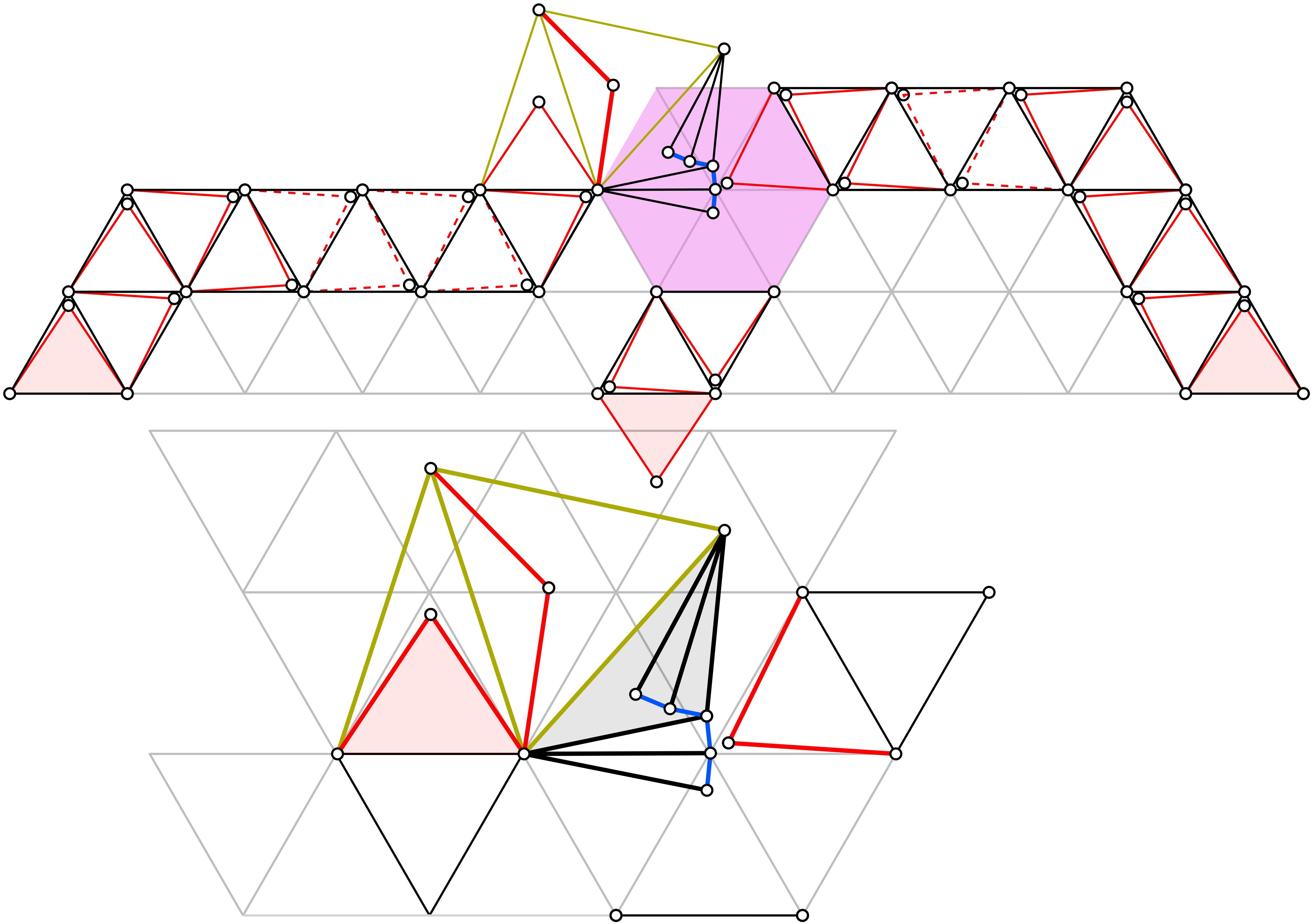
Clause gadget



Clause gadget



Clause gadget

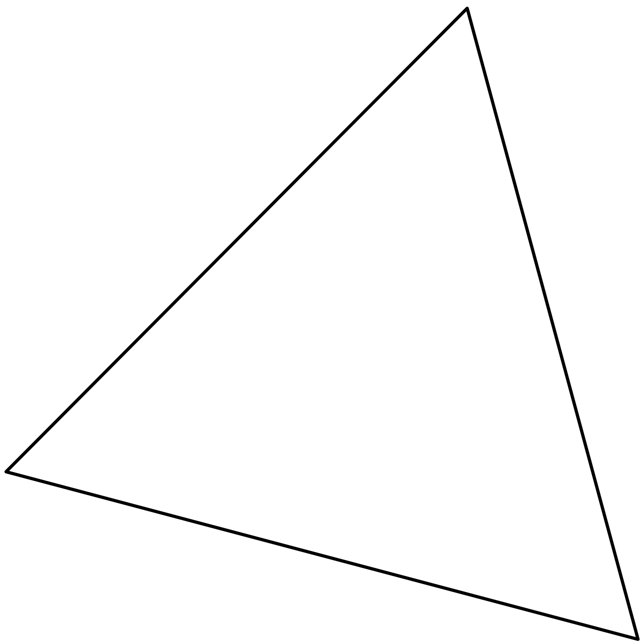


Theorem. The FEPR problem is NP-hard for weighted 2-trees, even for instances whose number of distinct edge lengths is 4.

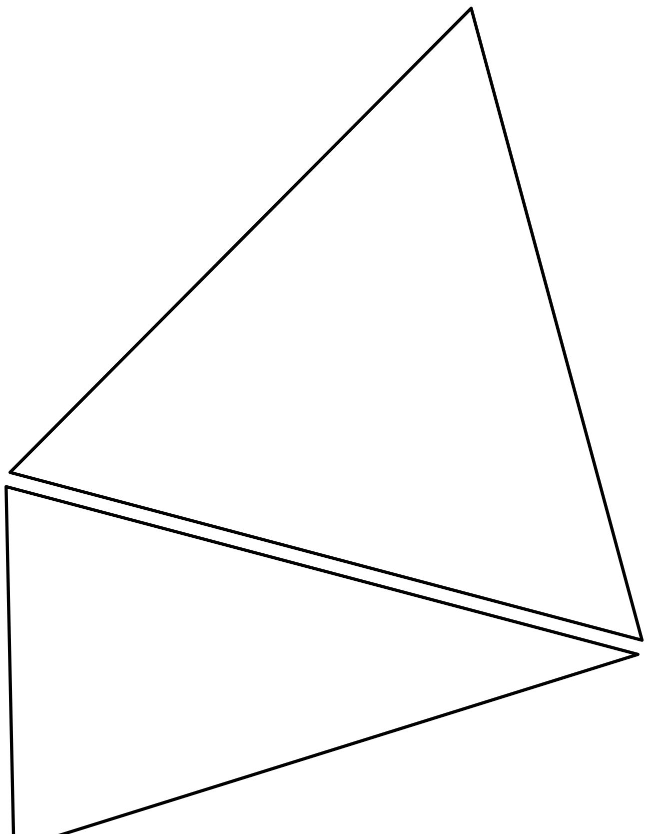
Two edge lenghts

Containment among triangles

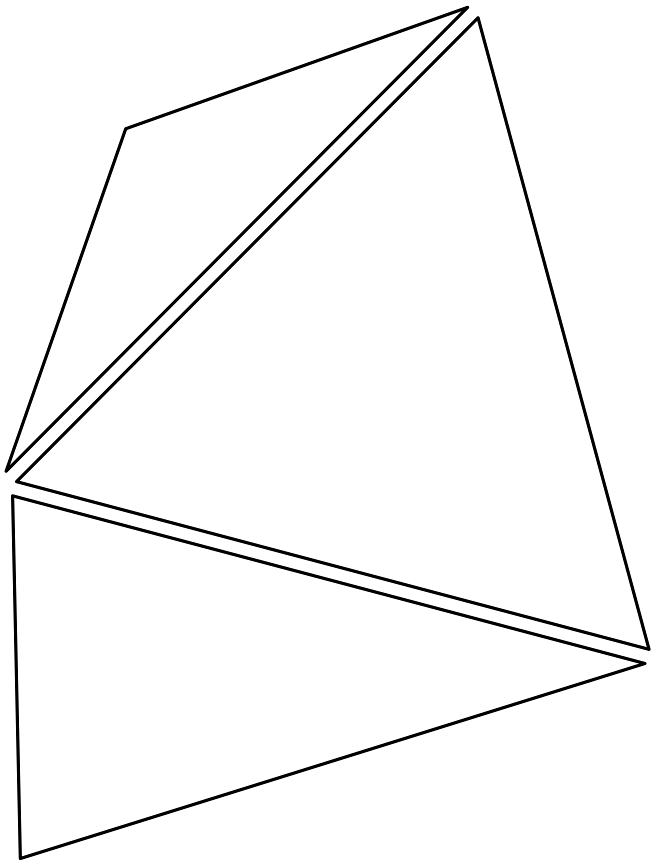
Containment among triangles



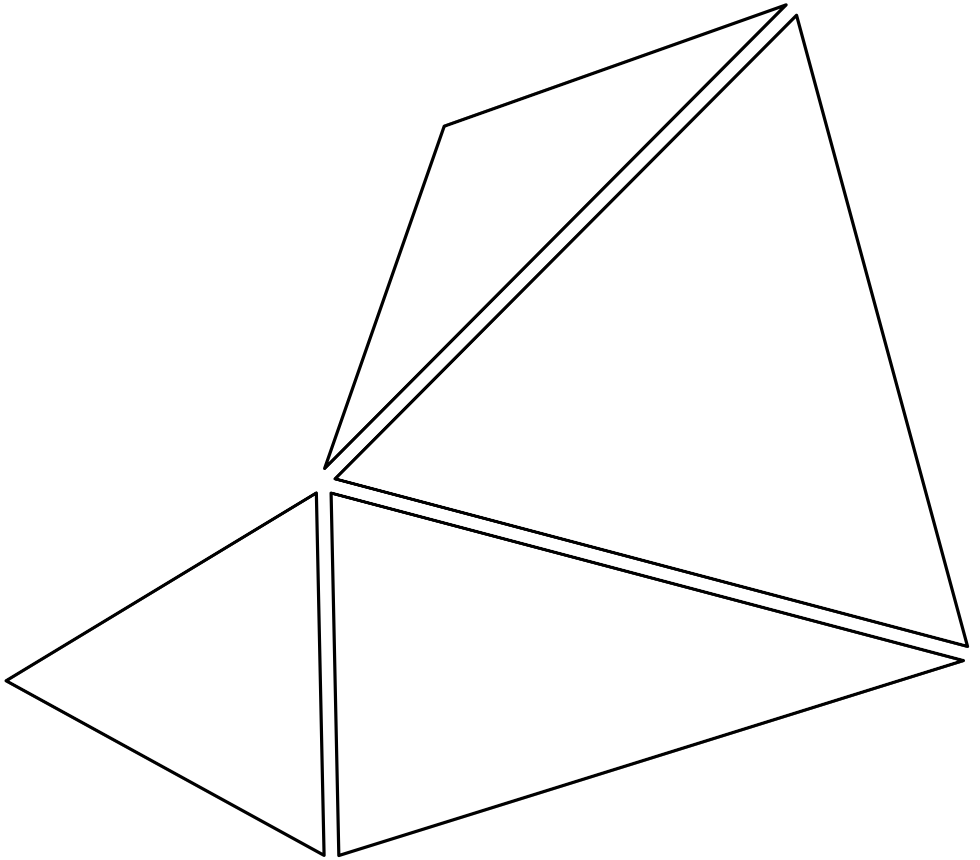
Containment among triangles



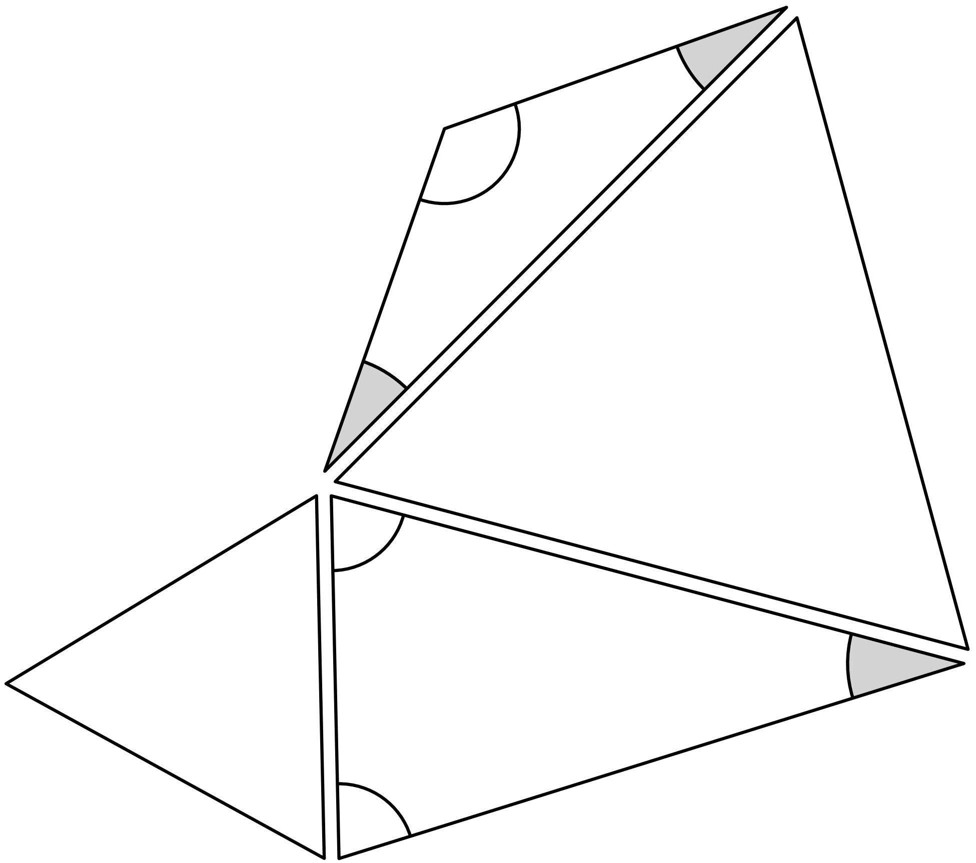
Containment among triangles



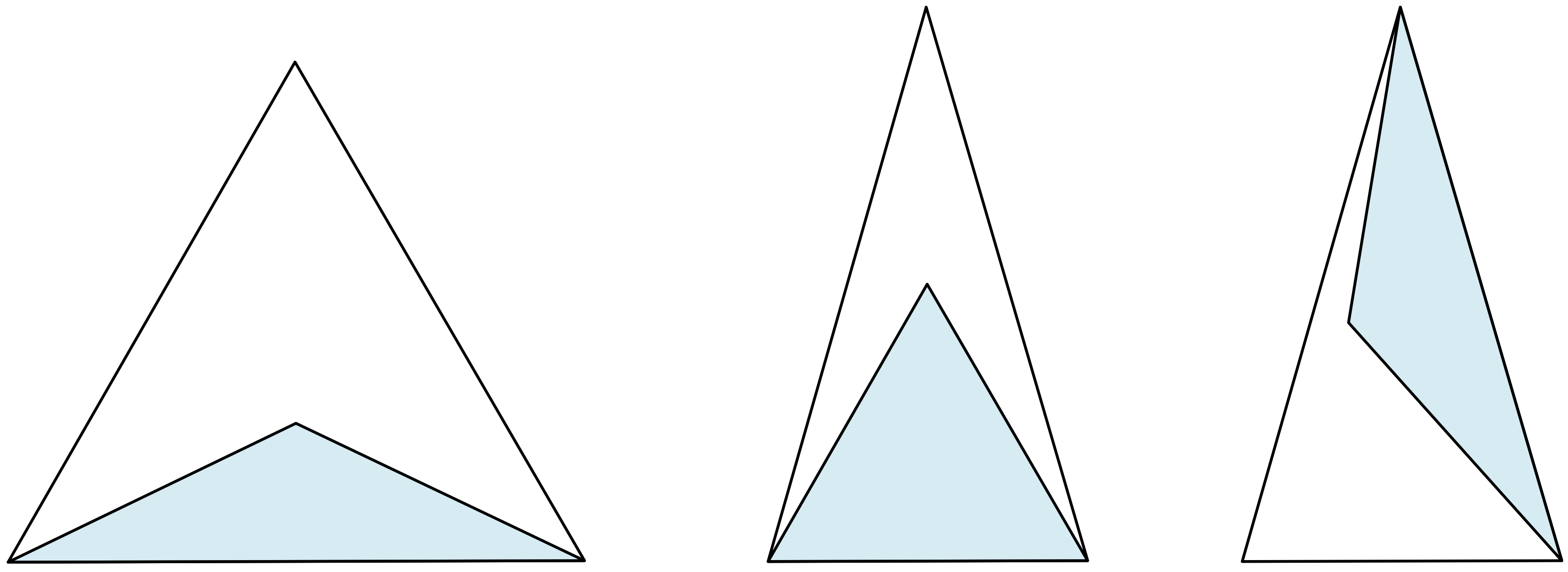
Containment among triangles



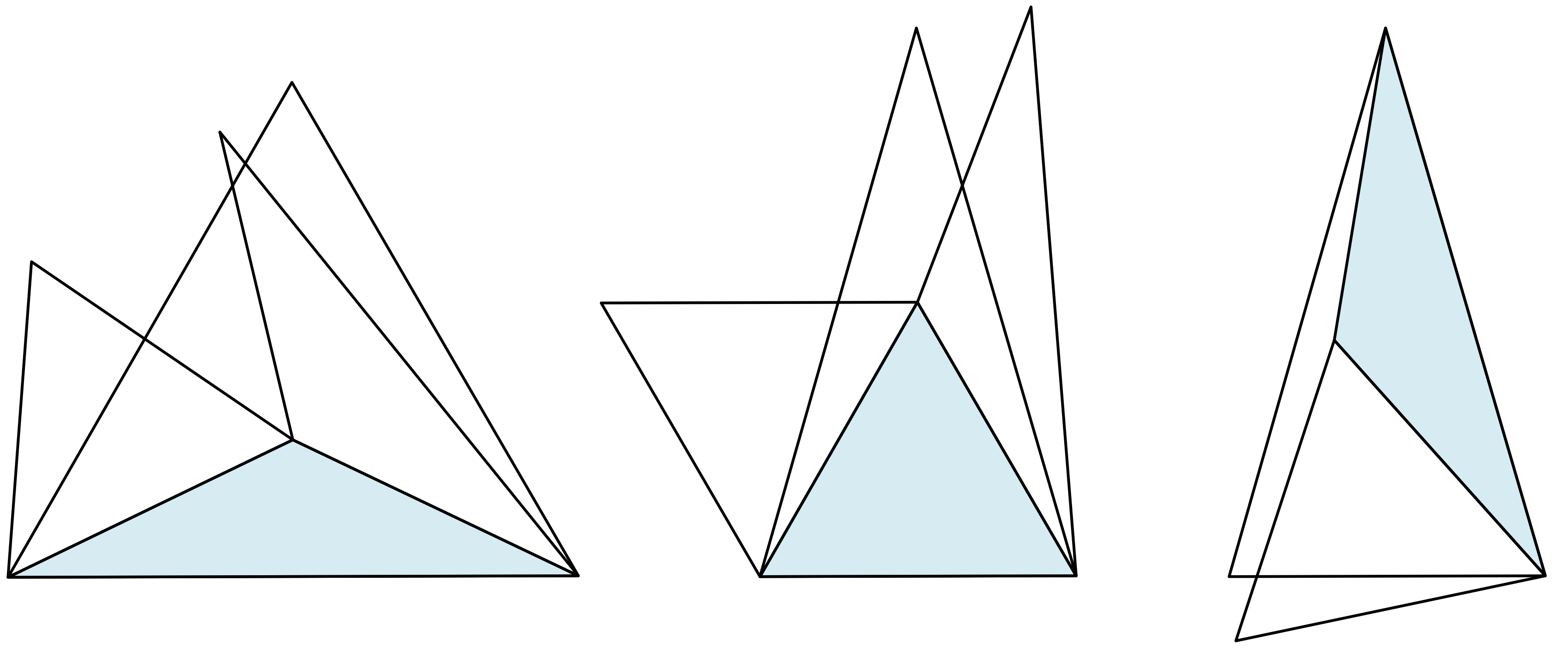
Containment among triangles



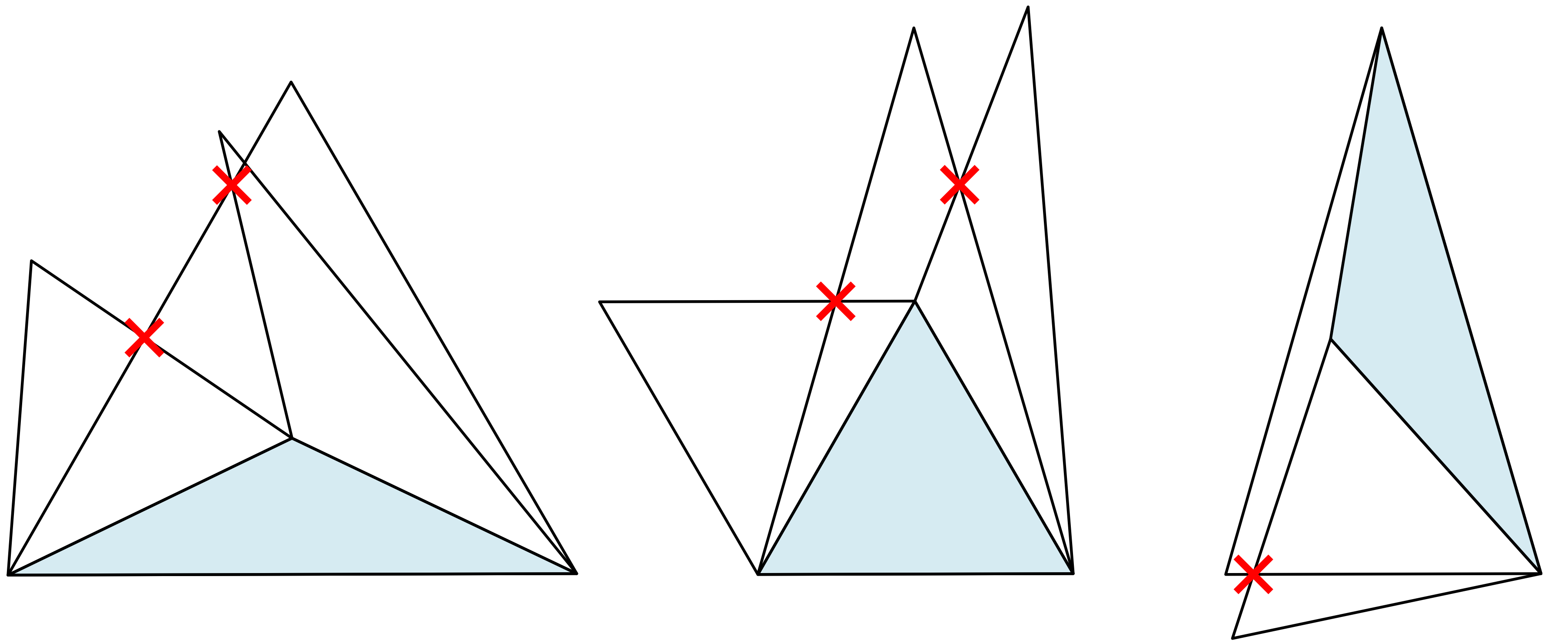
Containment among triangles



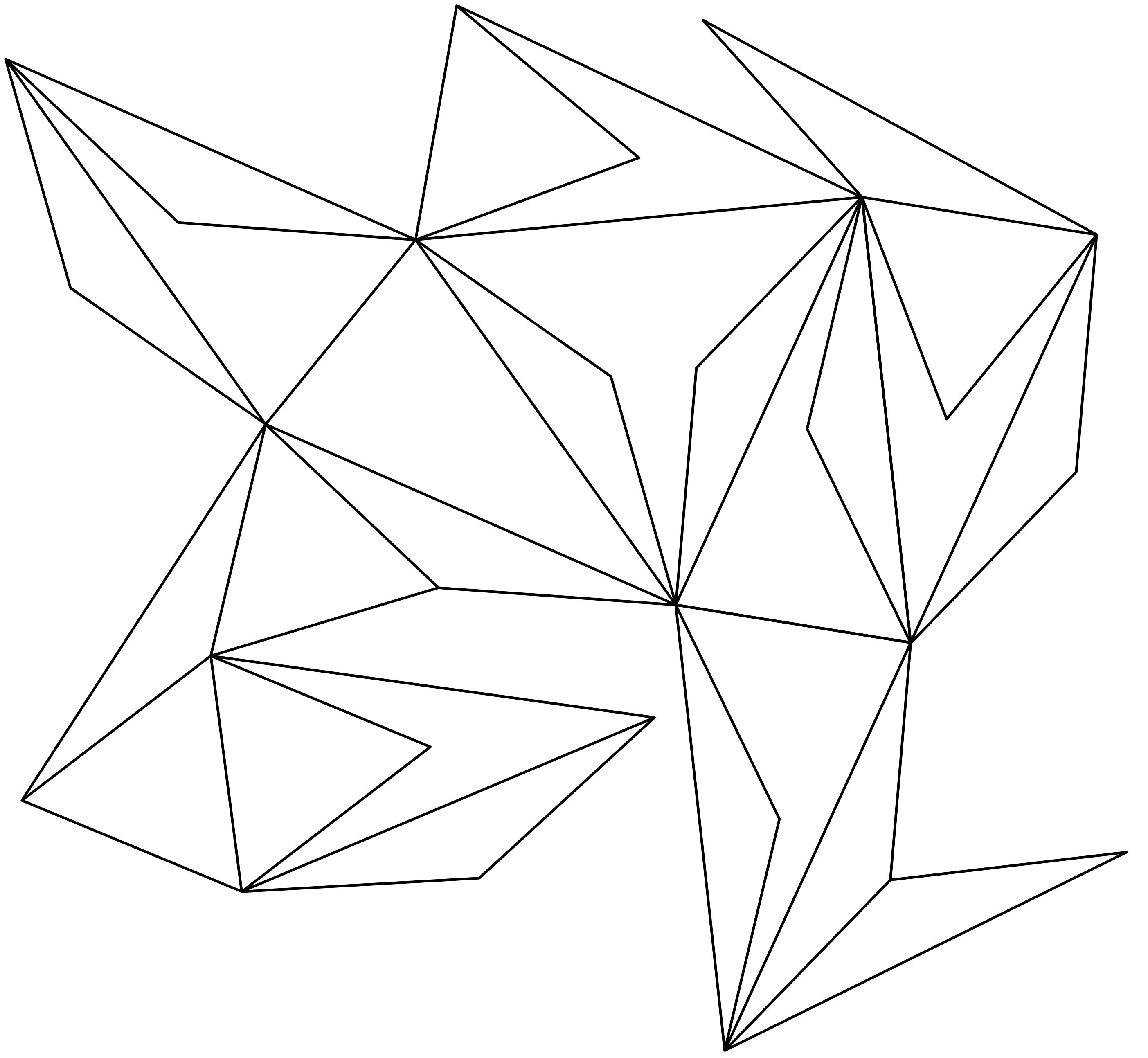
Containment among triangles



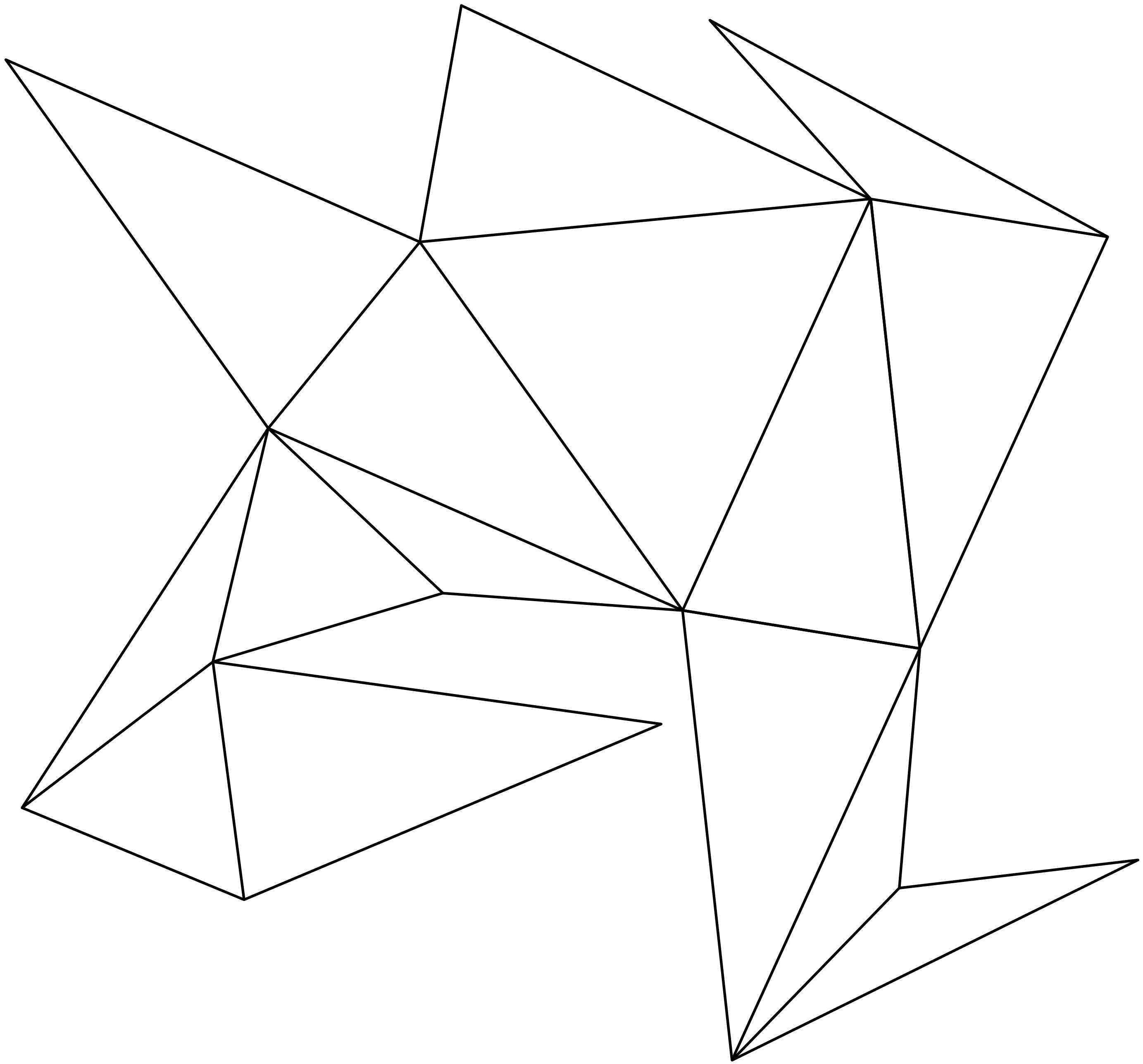
Containment among triangles



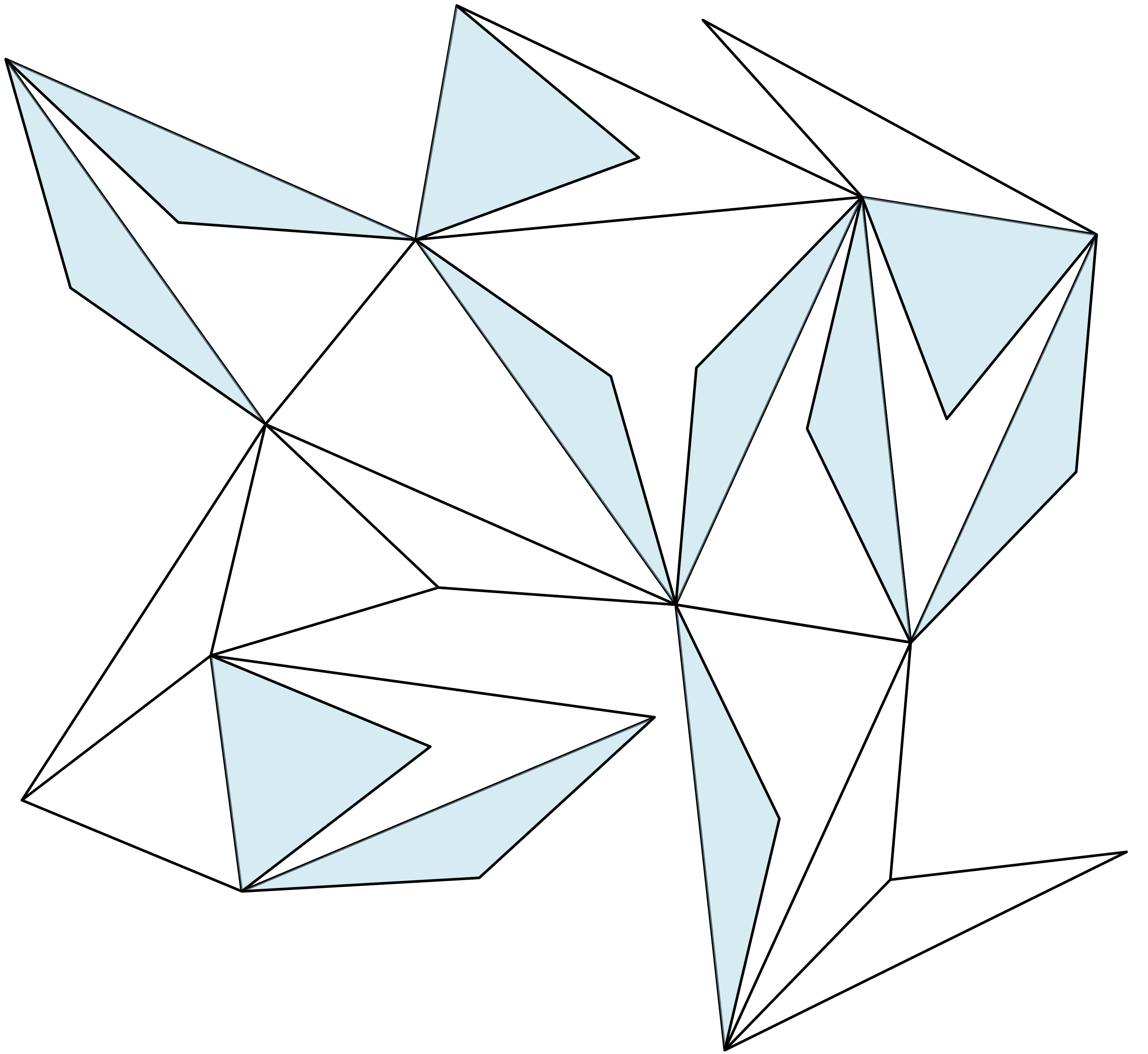
The frame and the set of leaf triangles



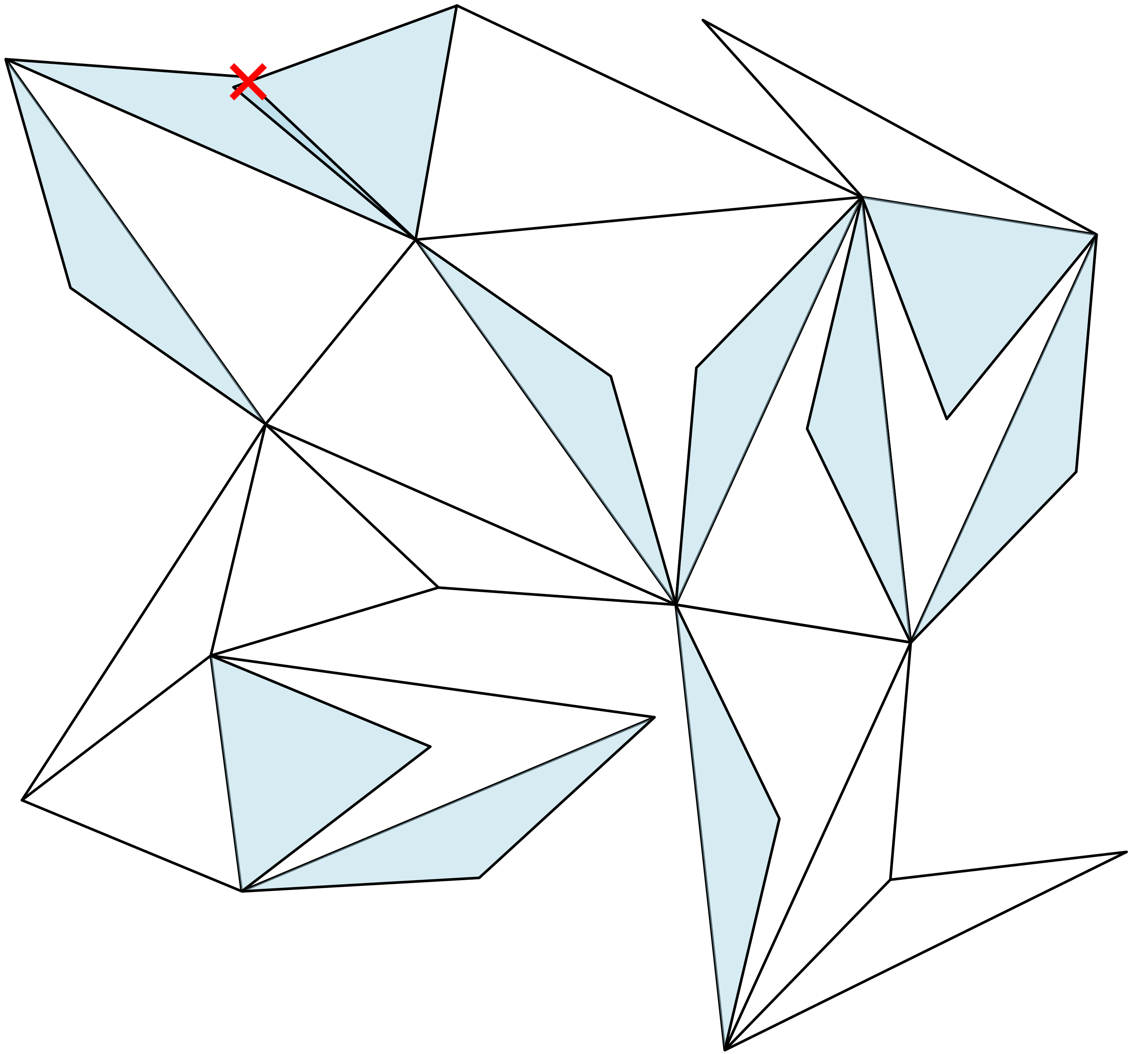
The frame and the set of leaf triangles



The frame and the set of leaf triangles

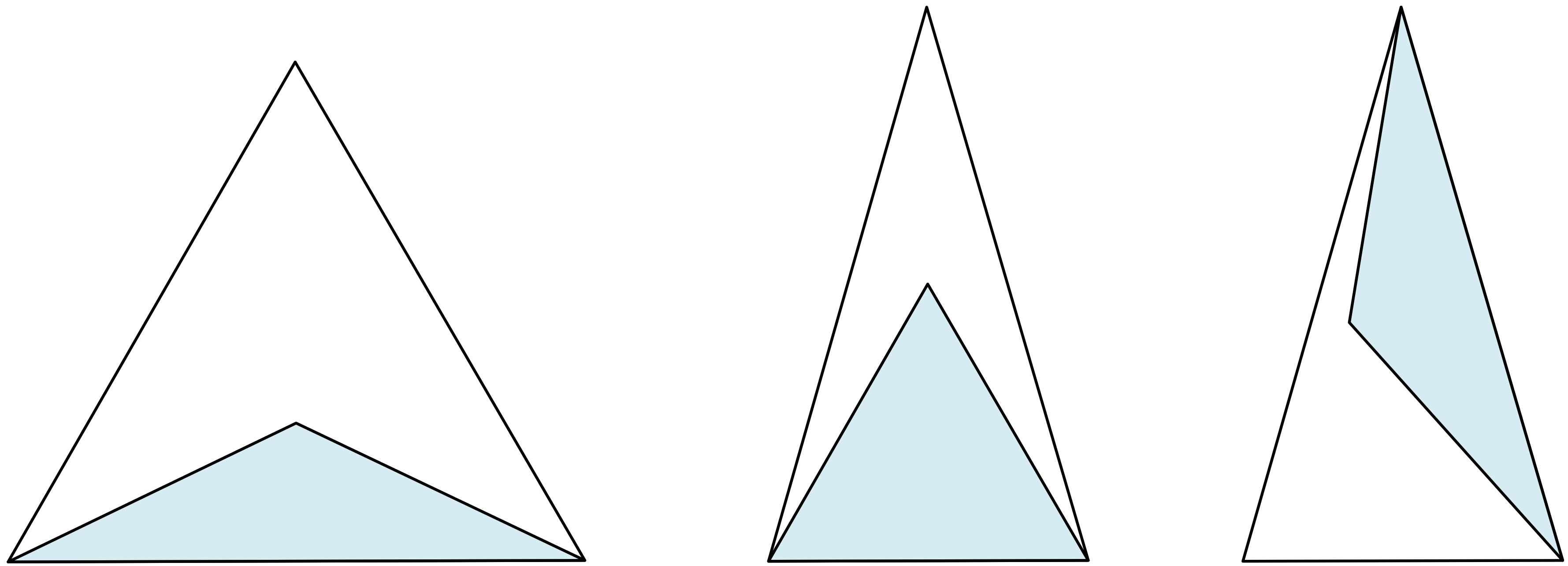


The frame and the set of leaf triangles

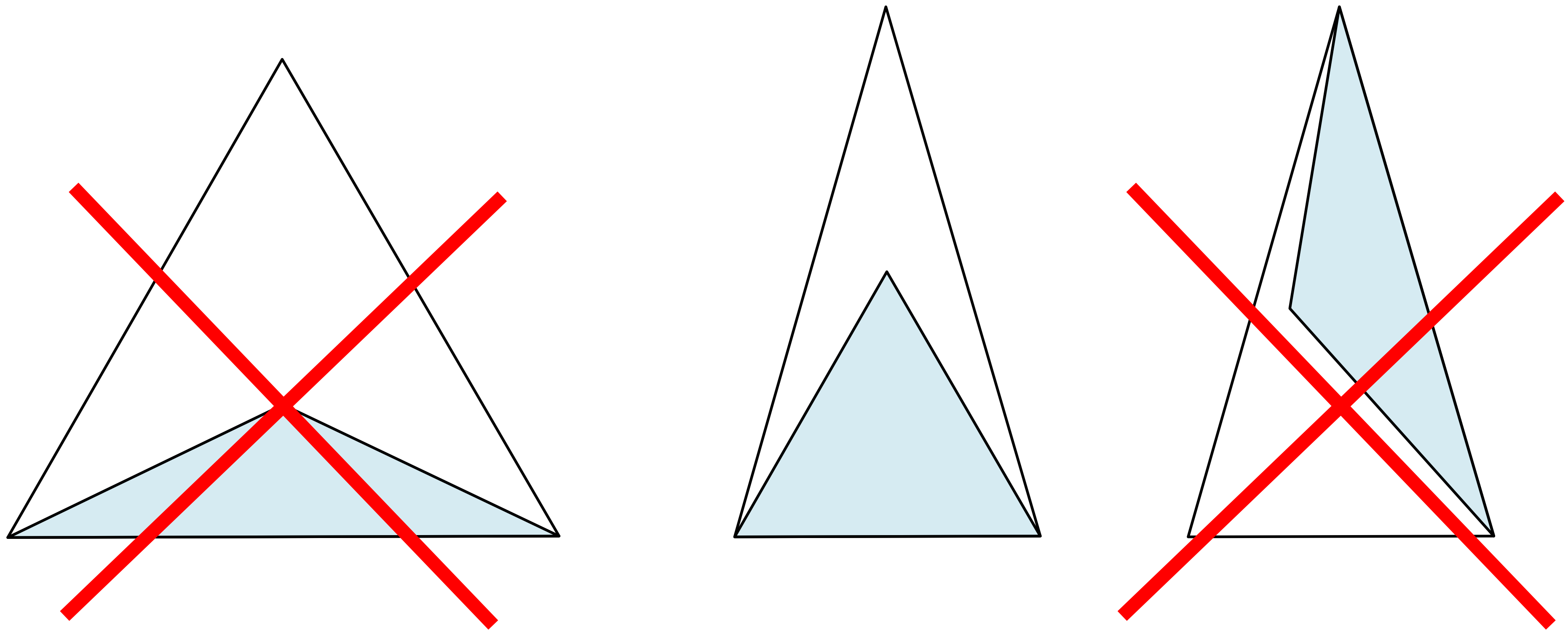


Ratio less than two

Ratio less than two

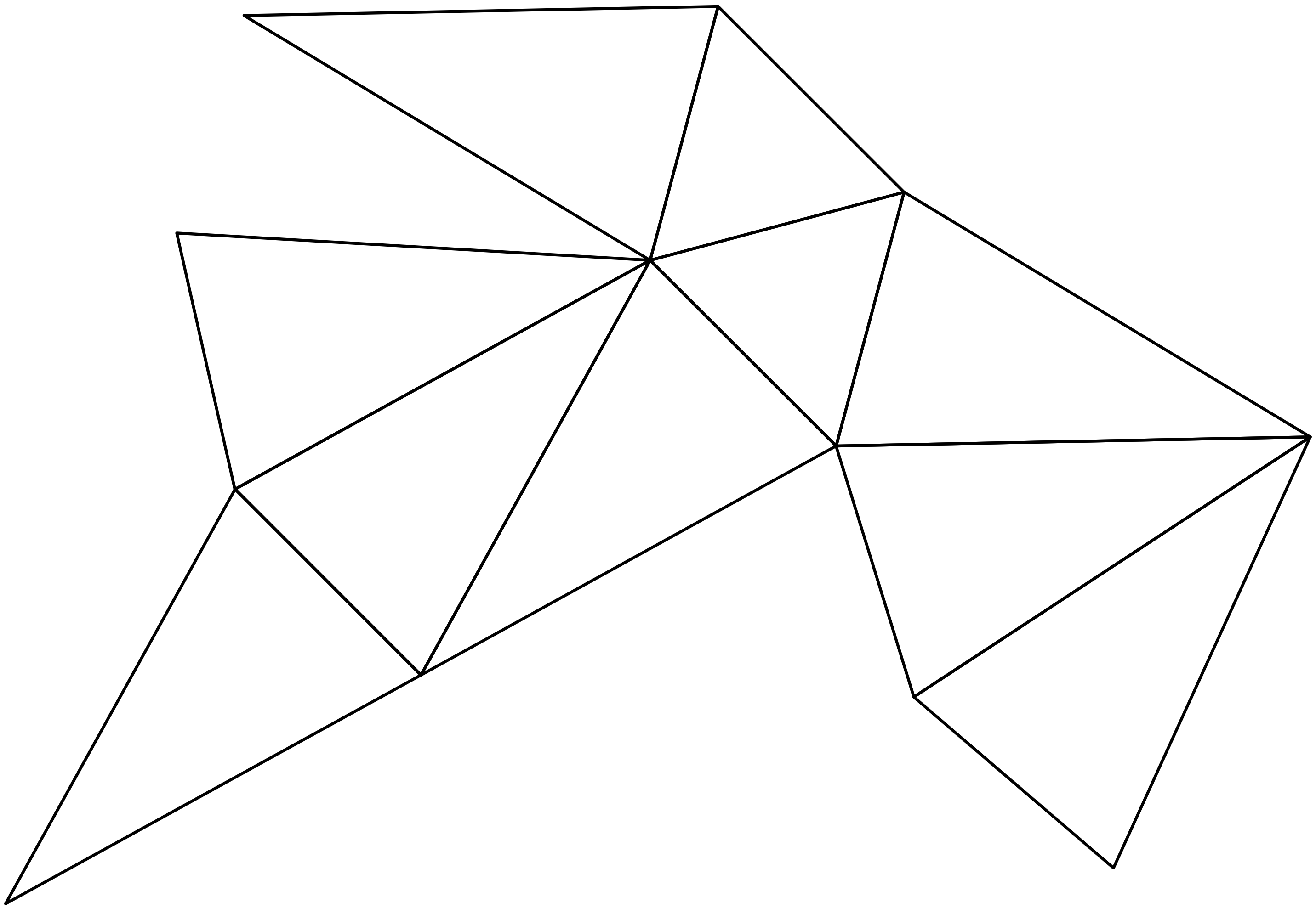


Ratio less than two

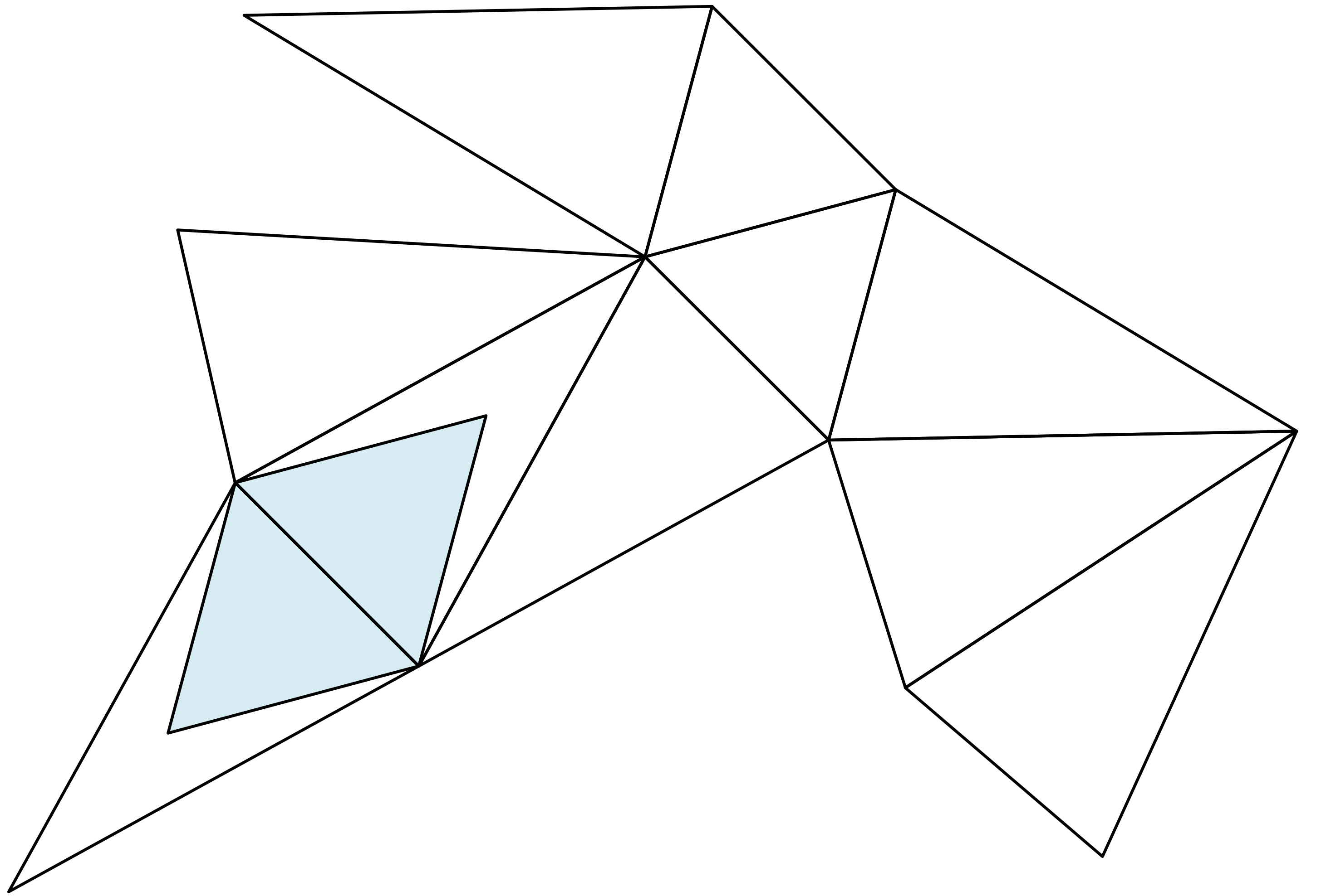


Ratio less than two

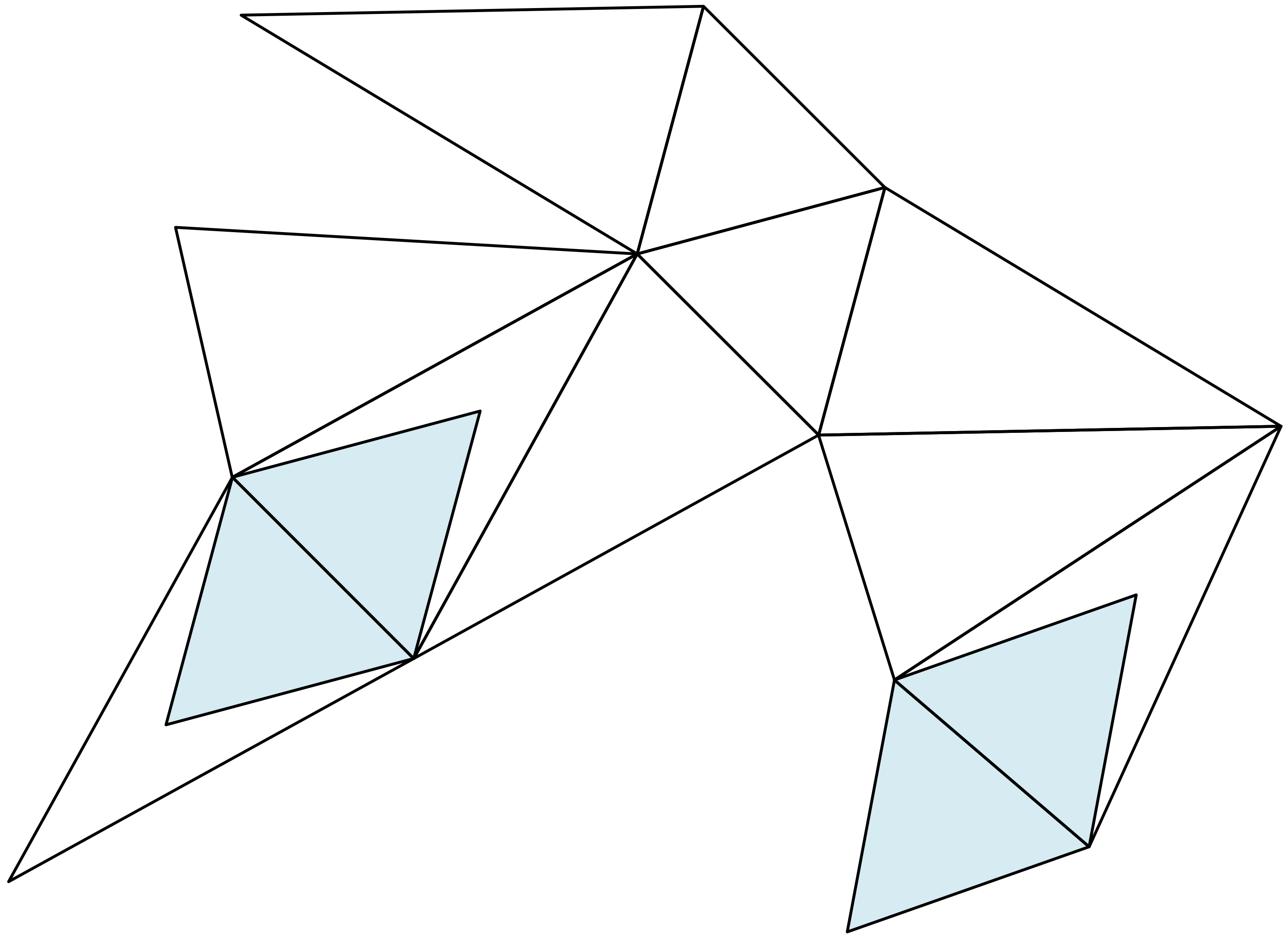
Ratio less than two



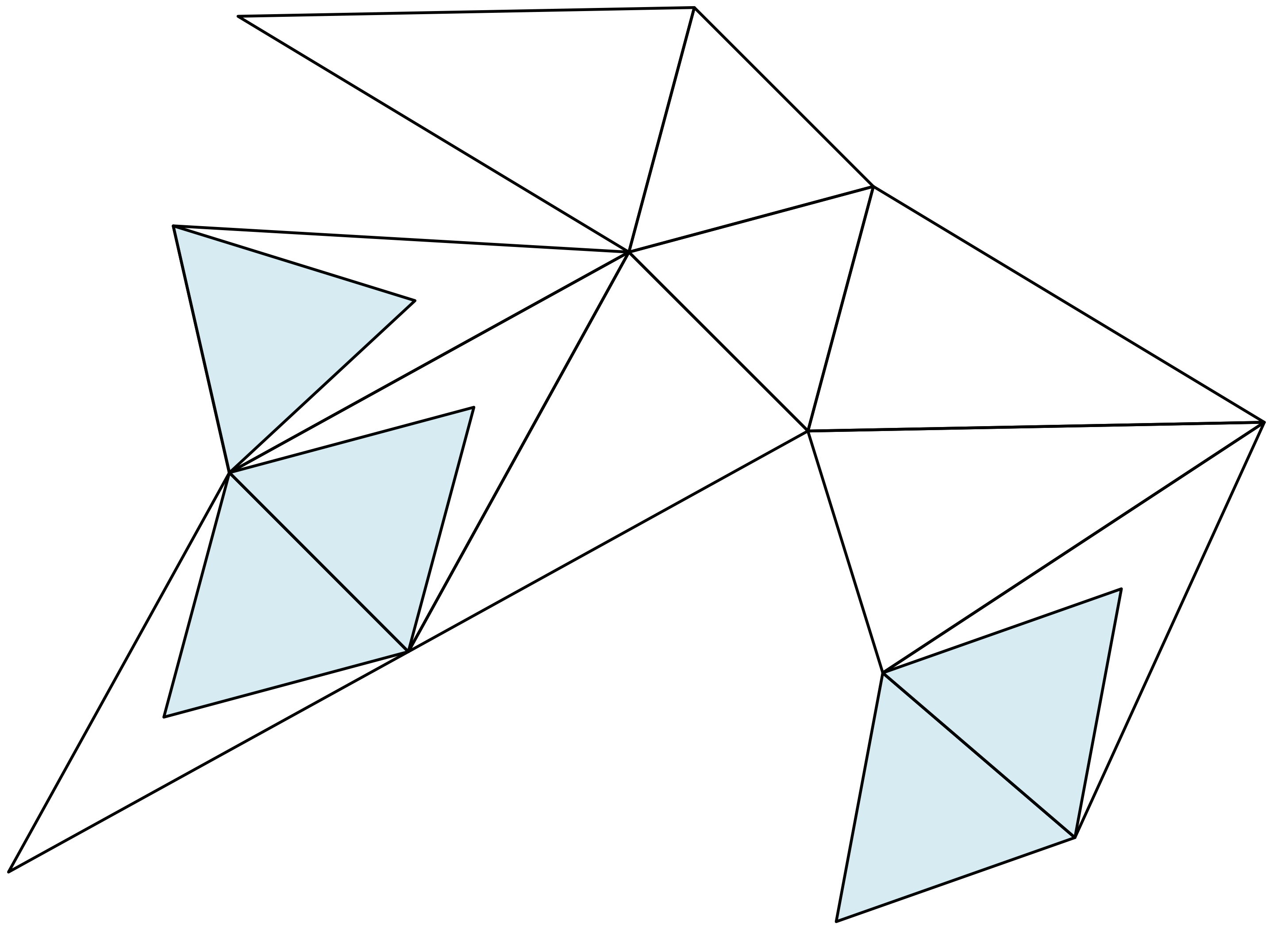
Ratio less than two



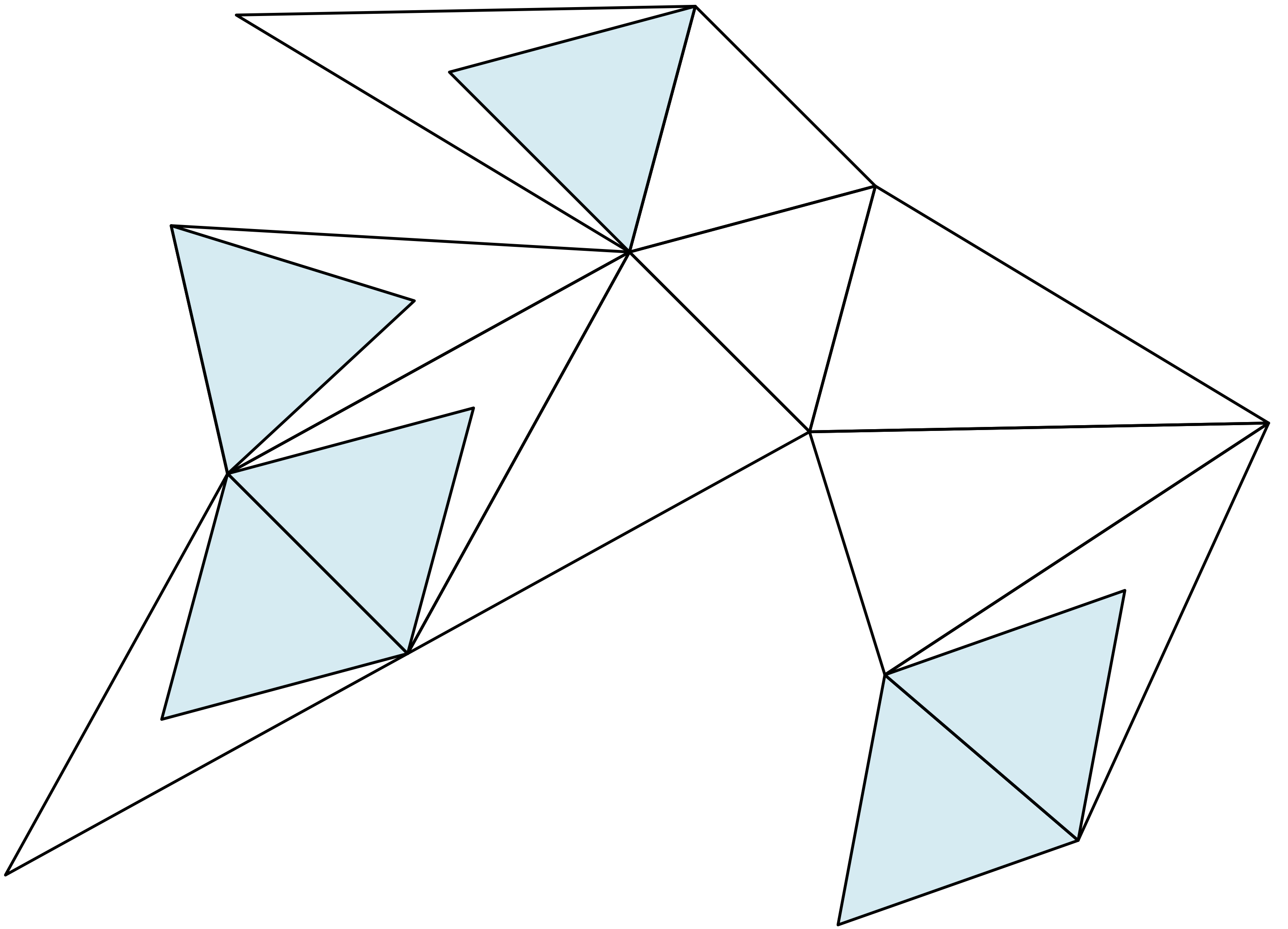
Ratio less than two



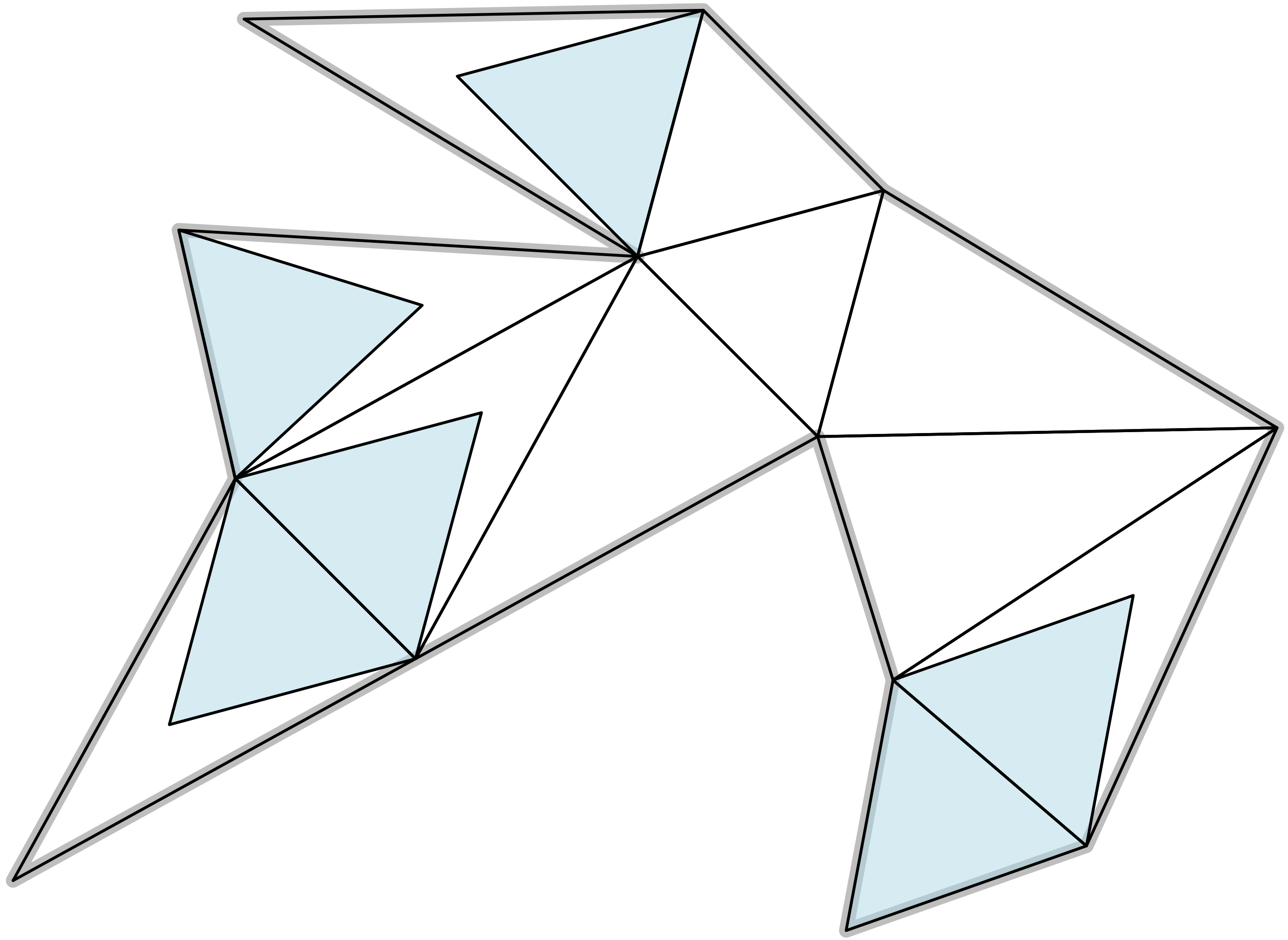
Ratio less than two



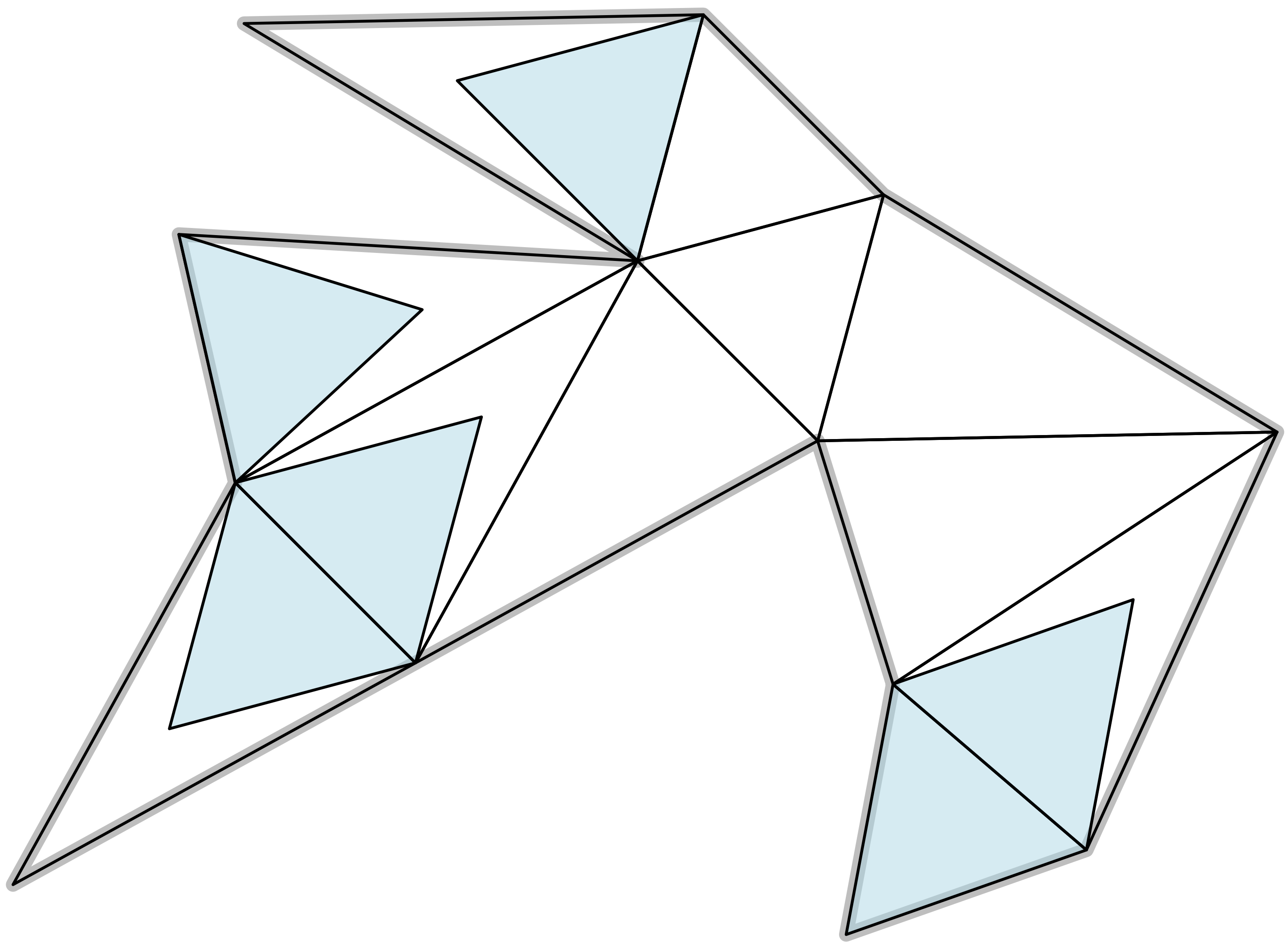
Ratio less than two



Ratio less than two

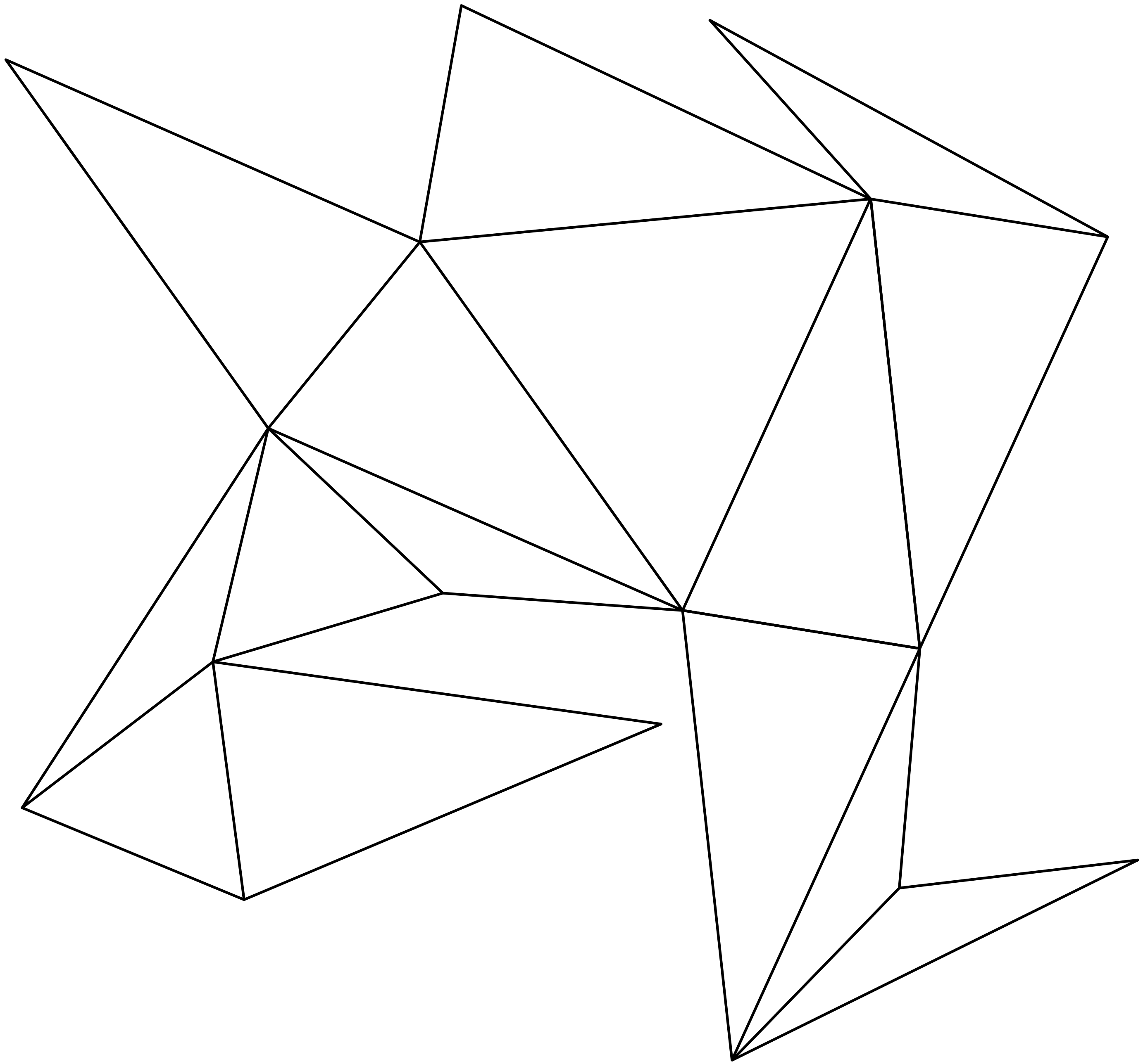


Ratio less than two

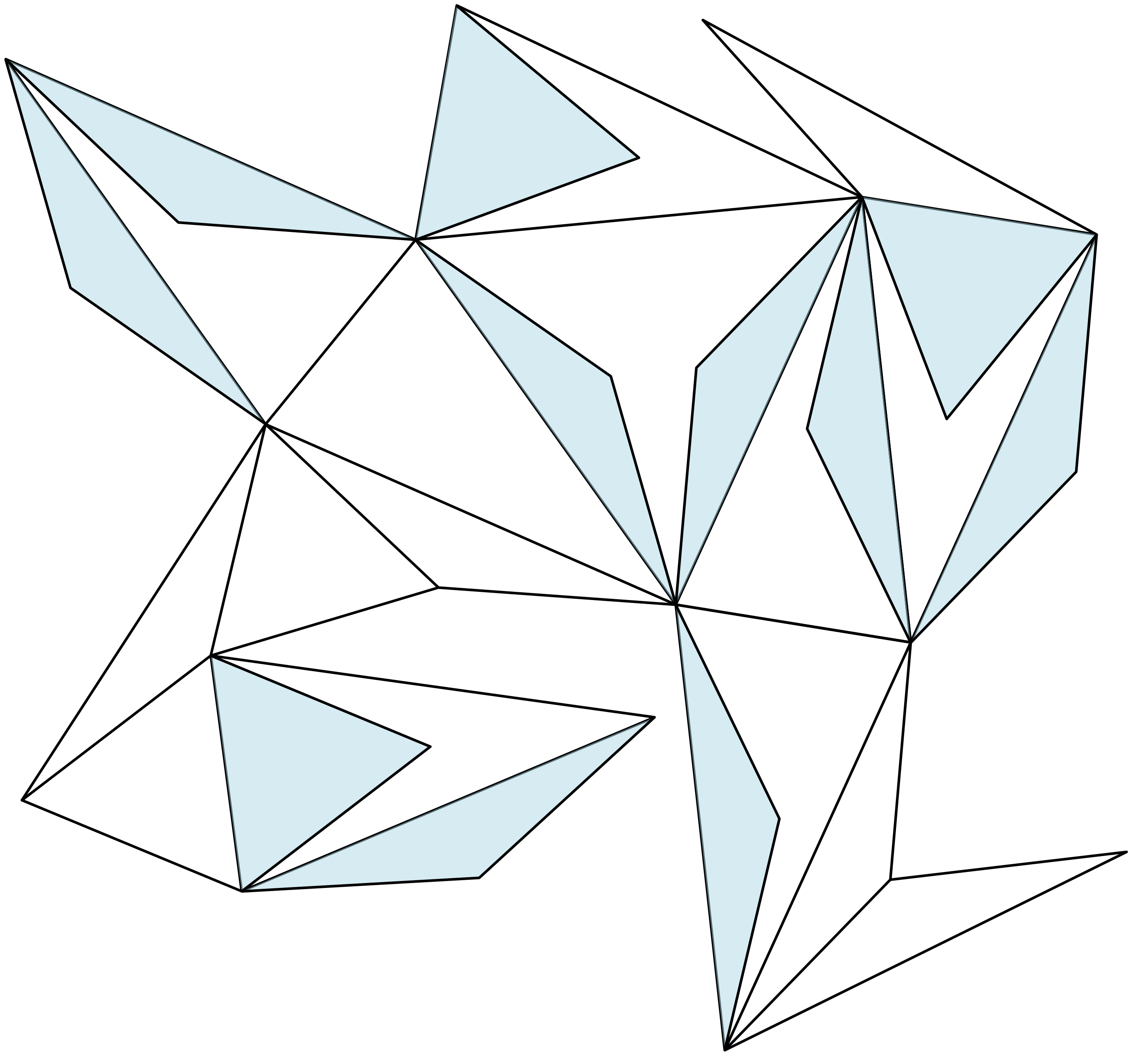


[Chazelle, 1991]
Triangulating a simple polygon in linear time

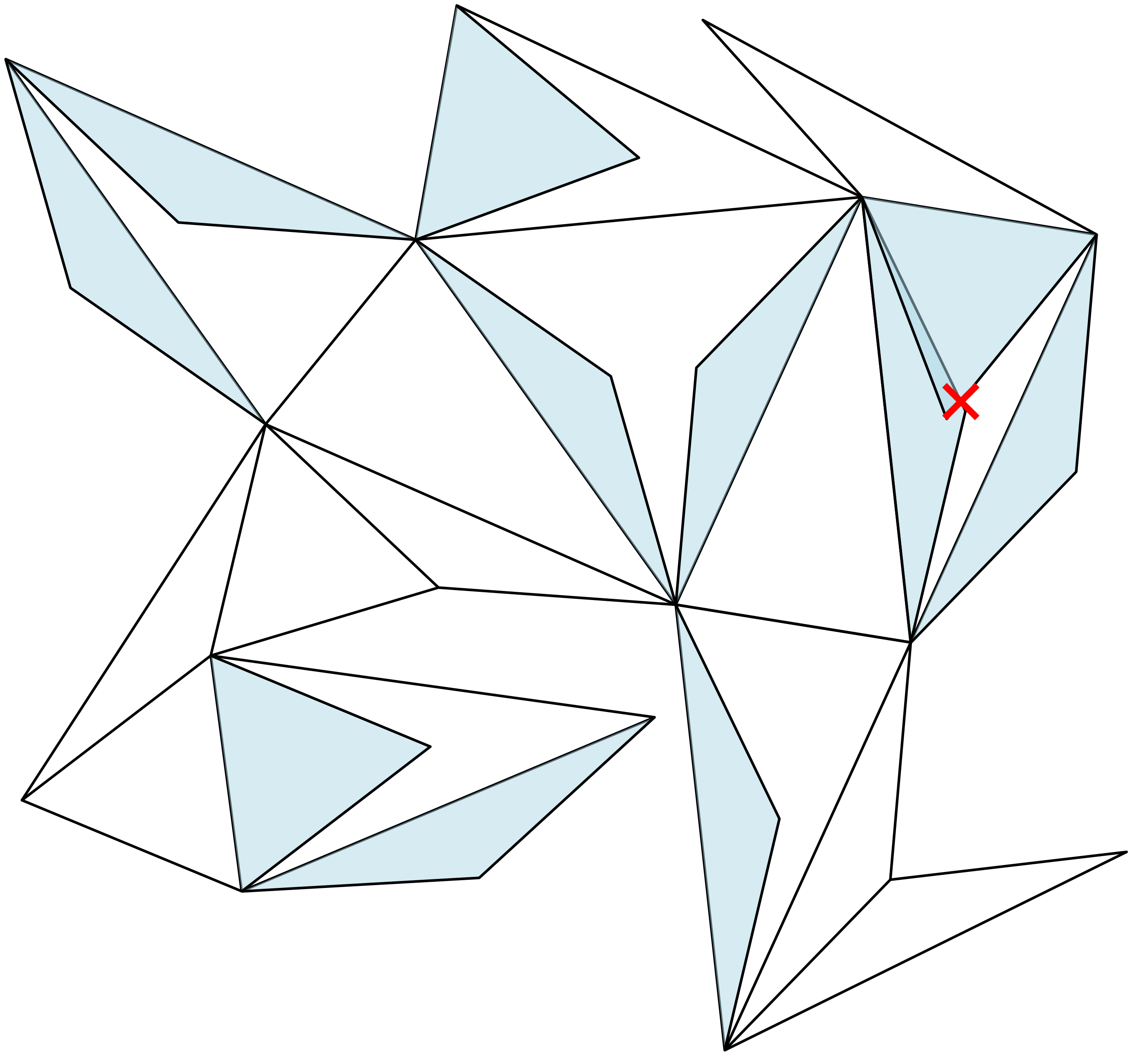
Ratio greater than two



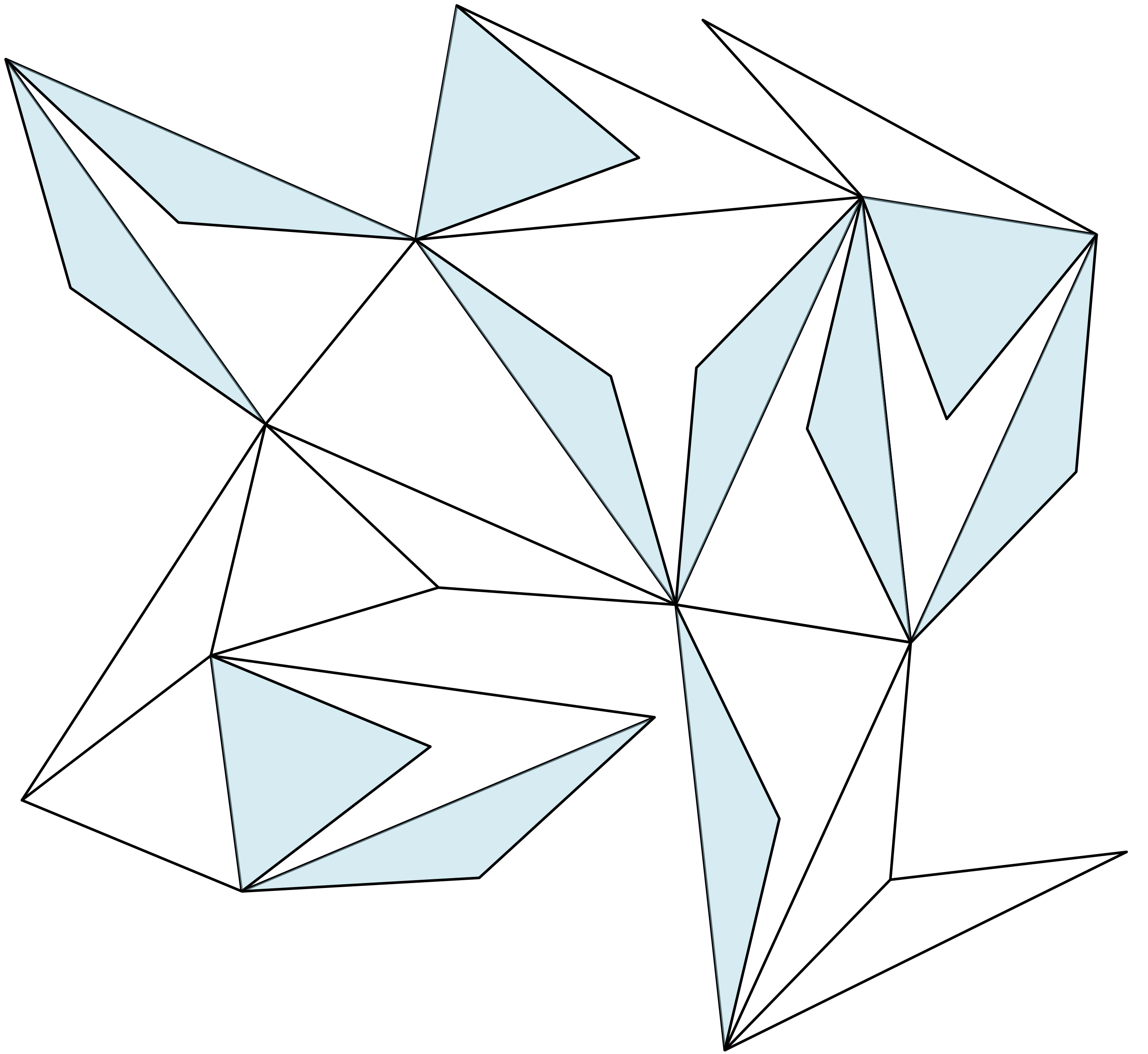
Ratio greater than two



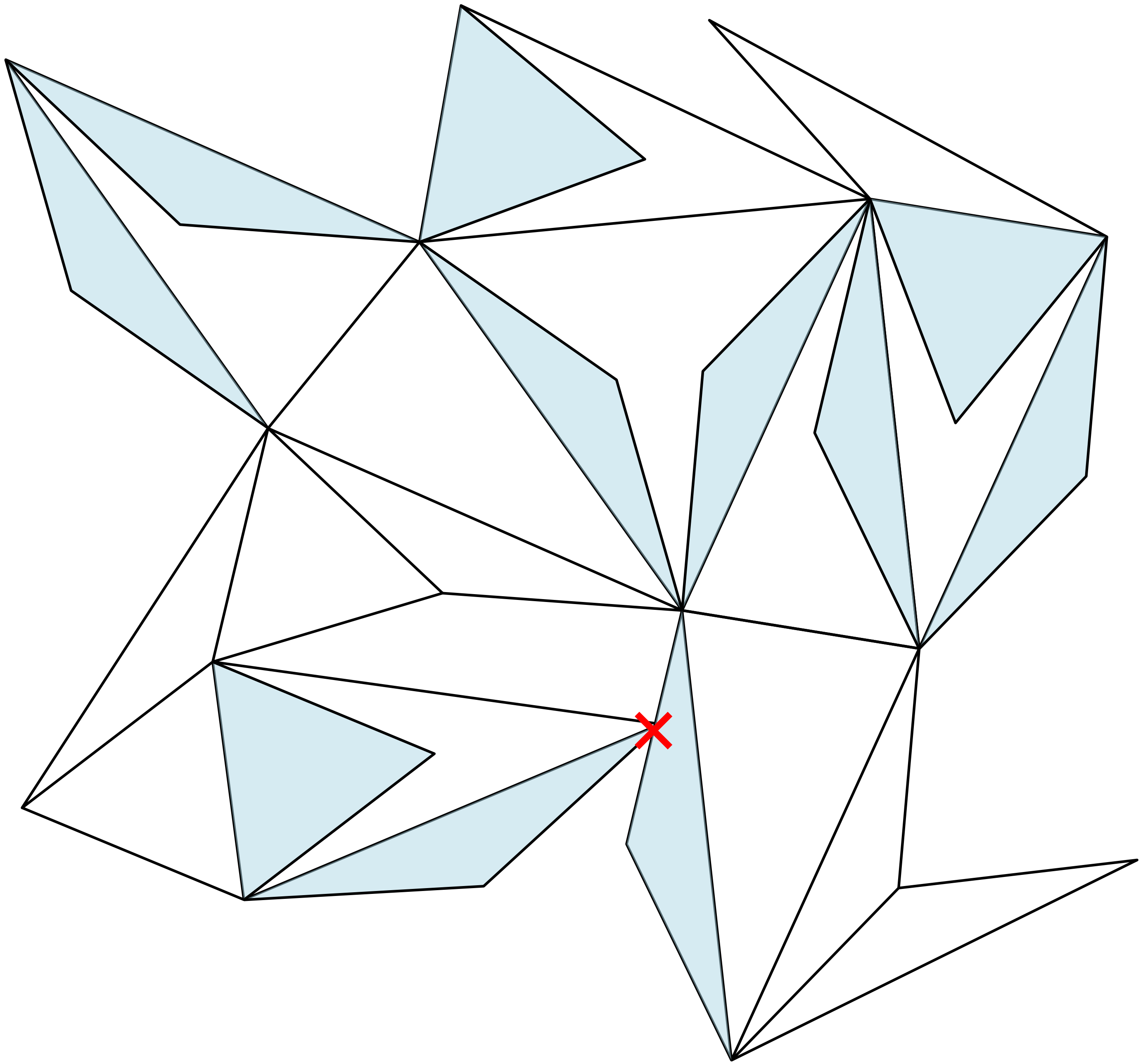
Ratio greater than two



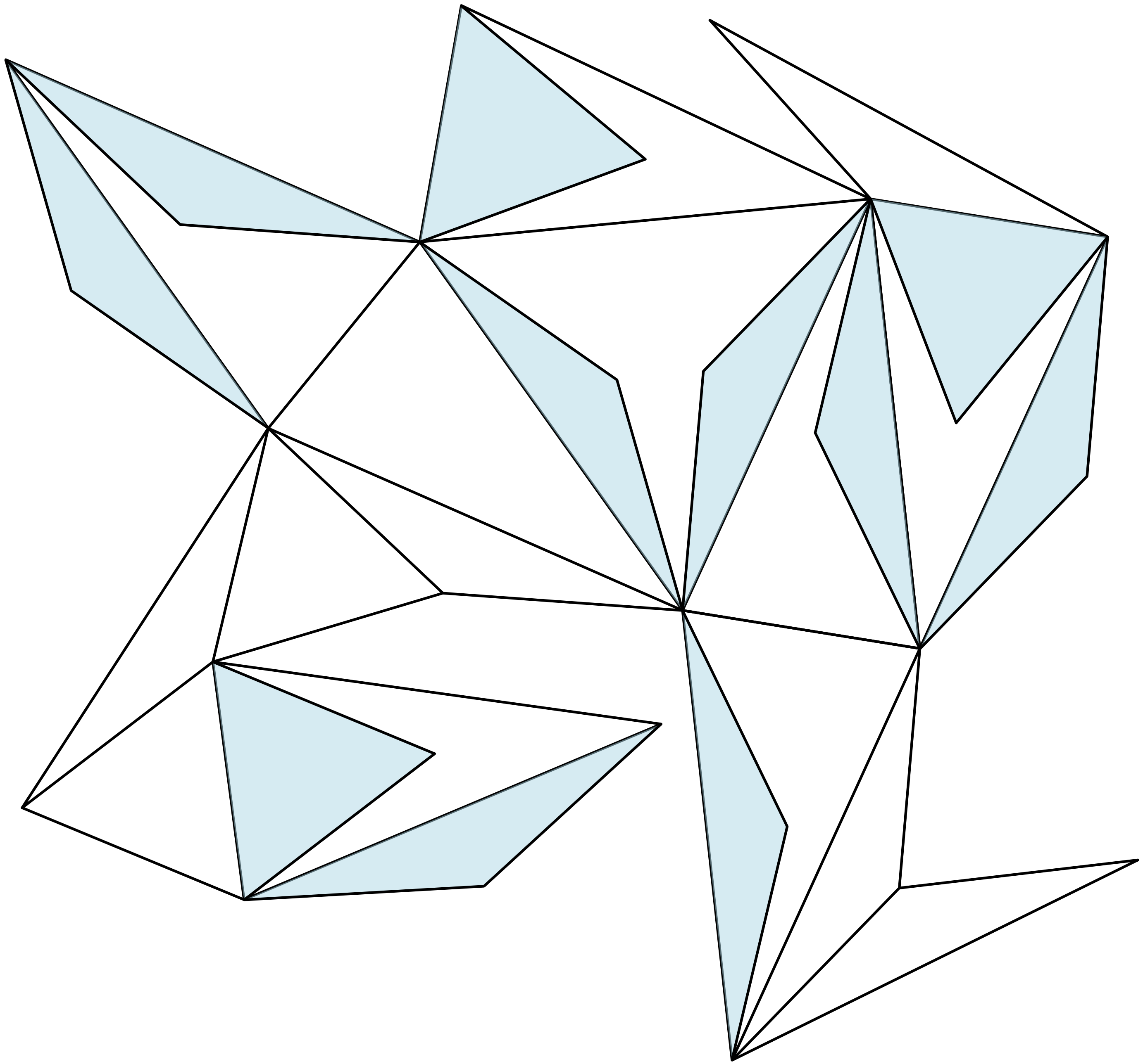
Ratio greater than two



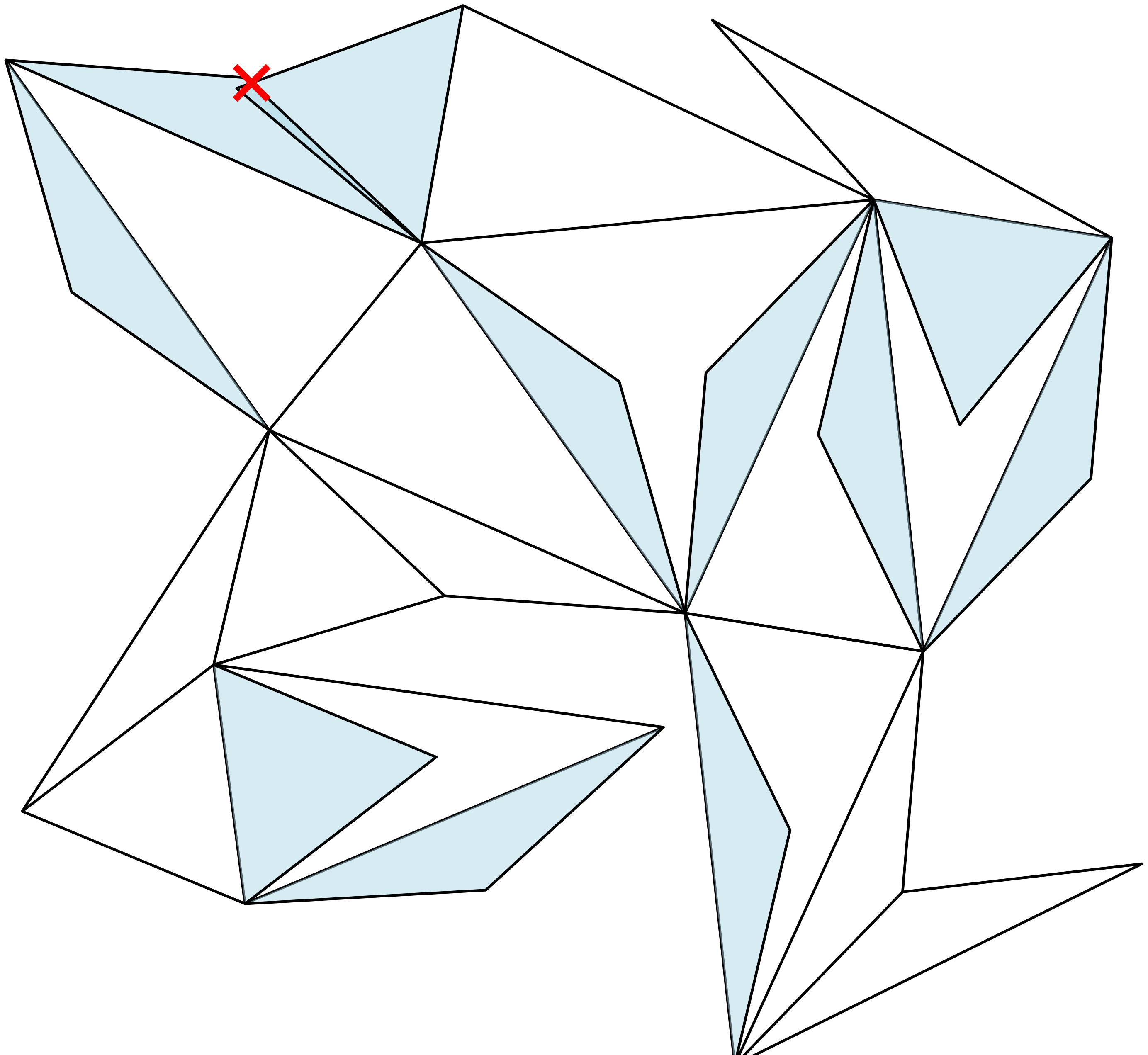
Ratio greater than two

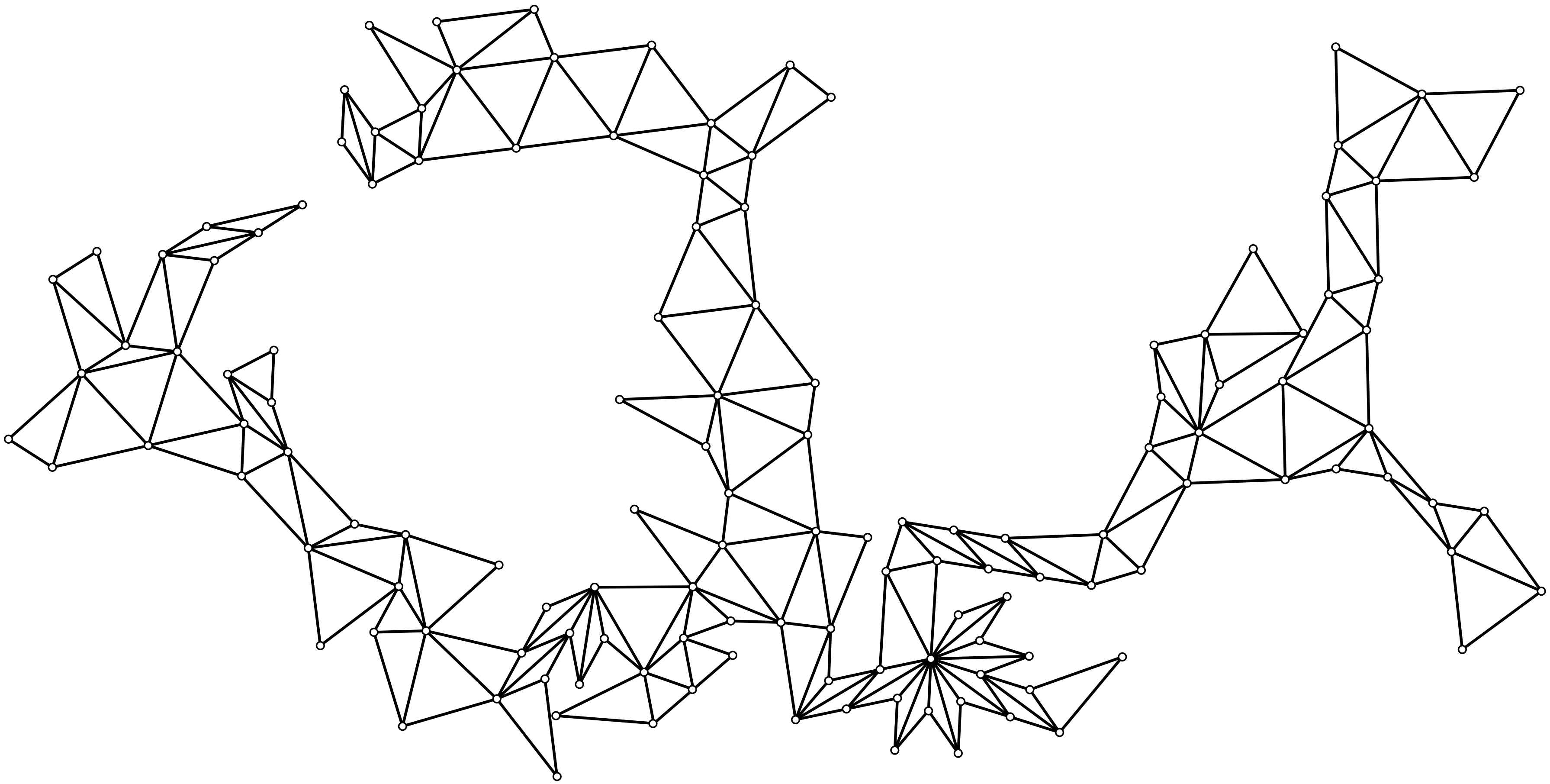


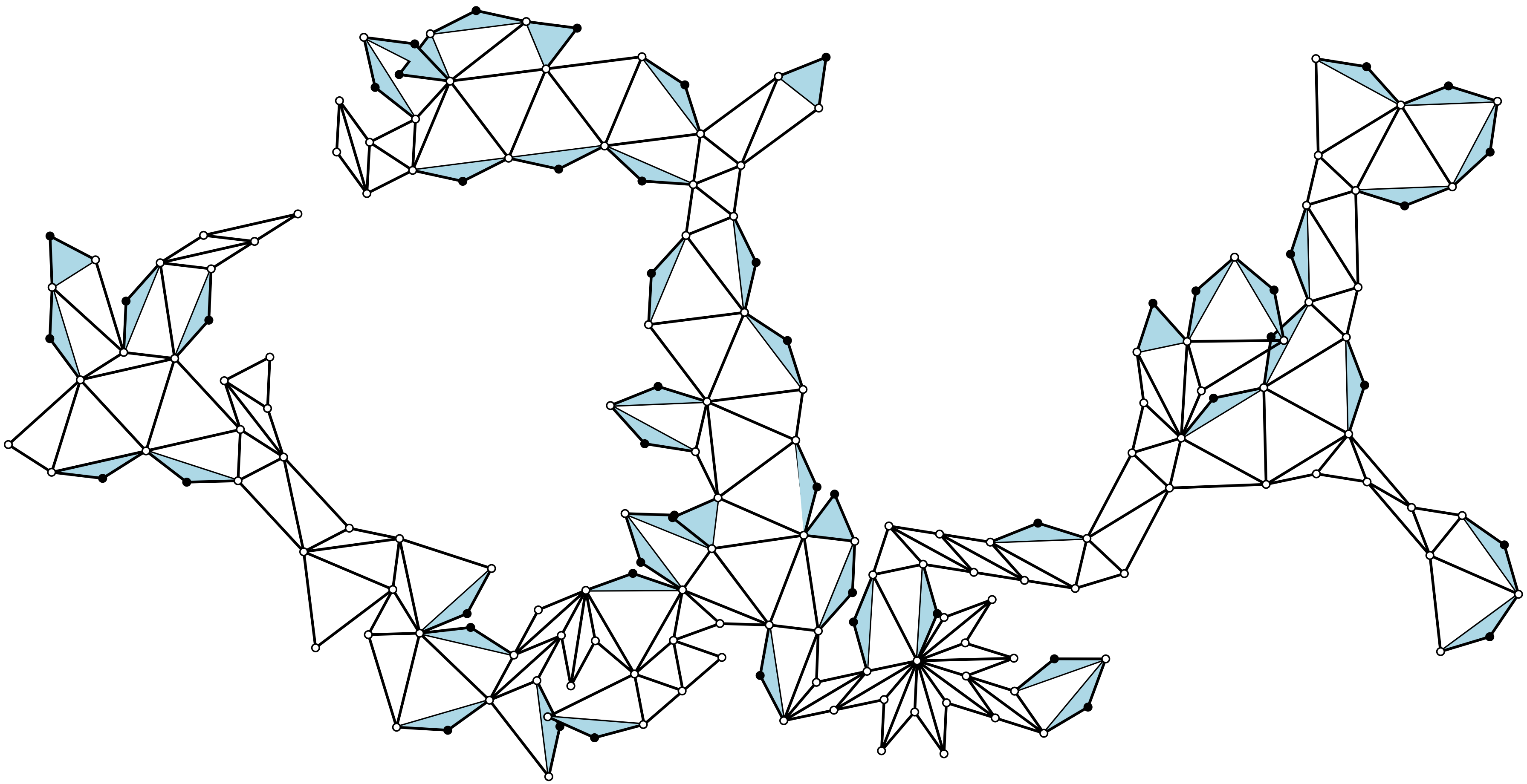
Ratio greater than two

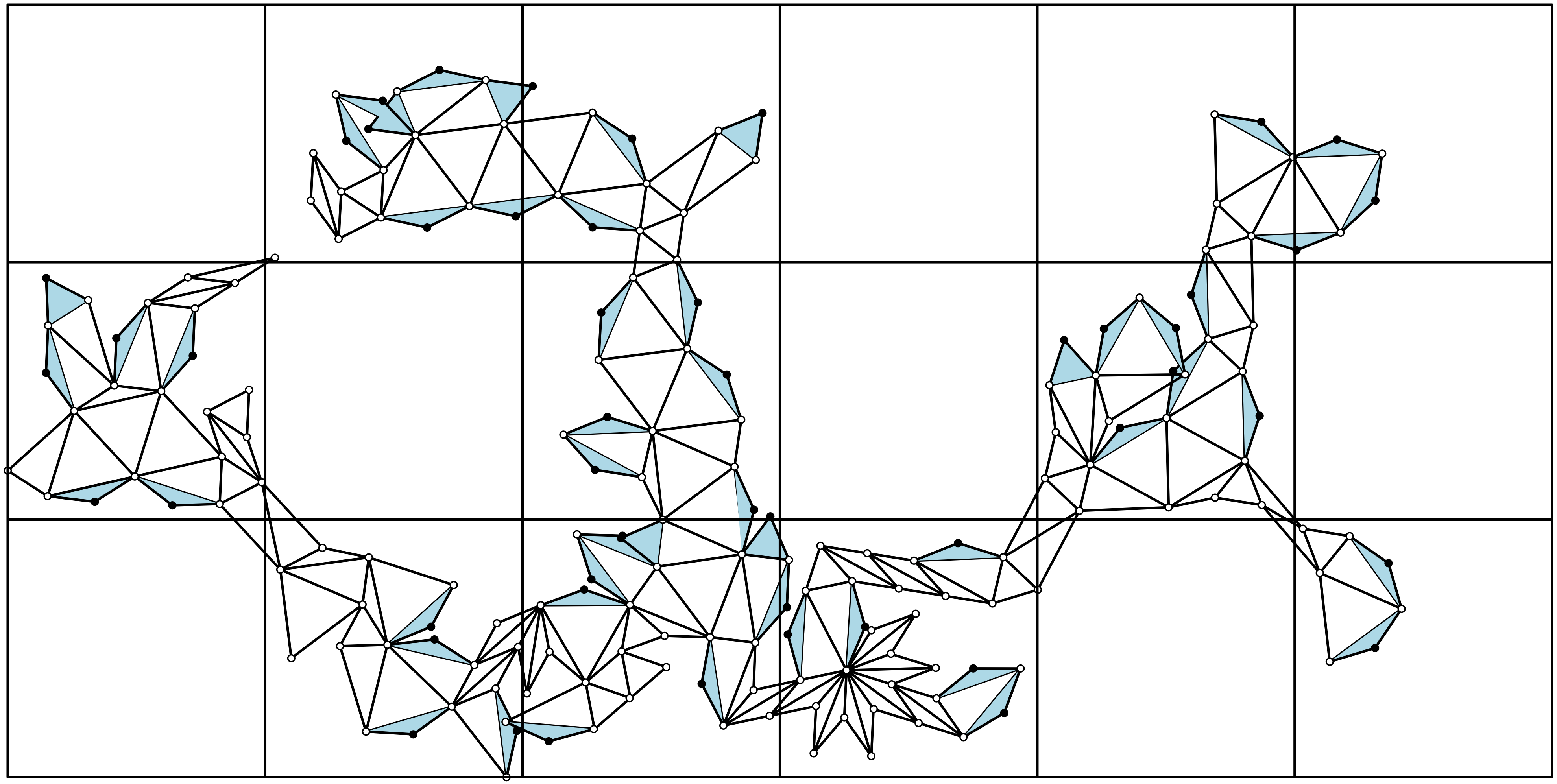


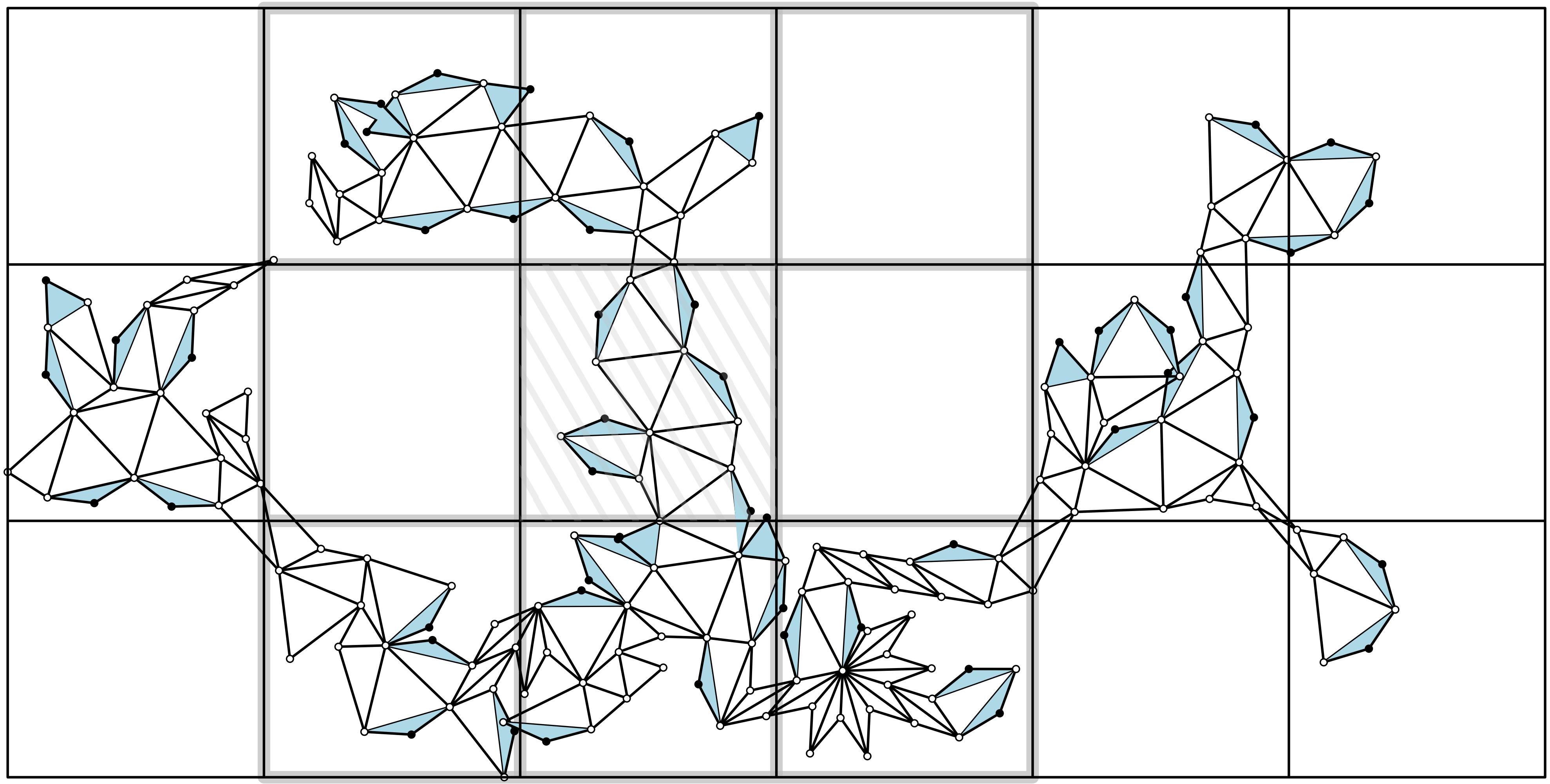
Ratio greater than two

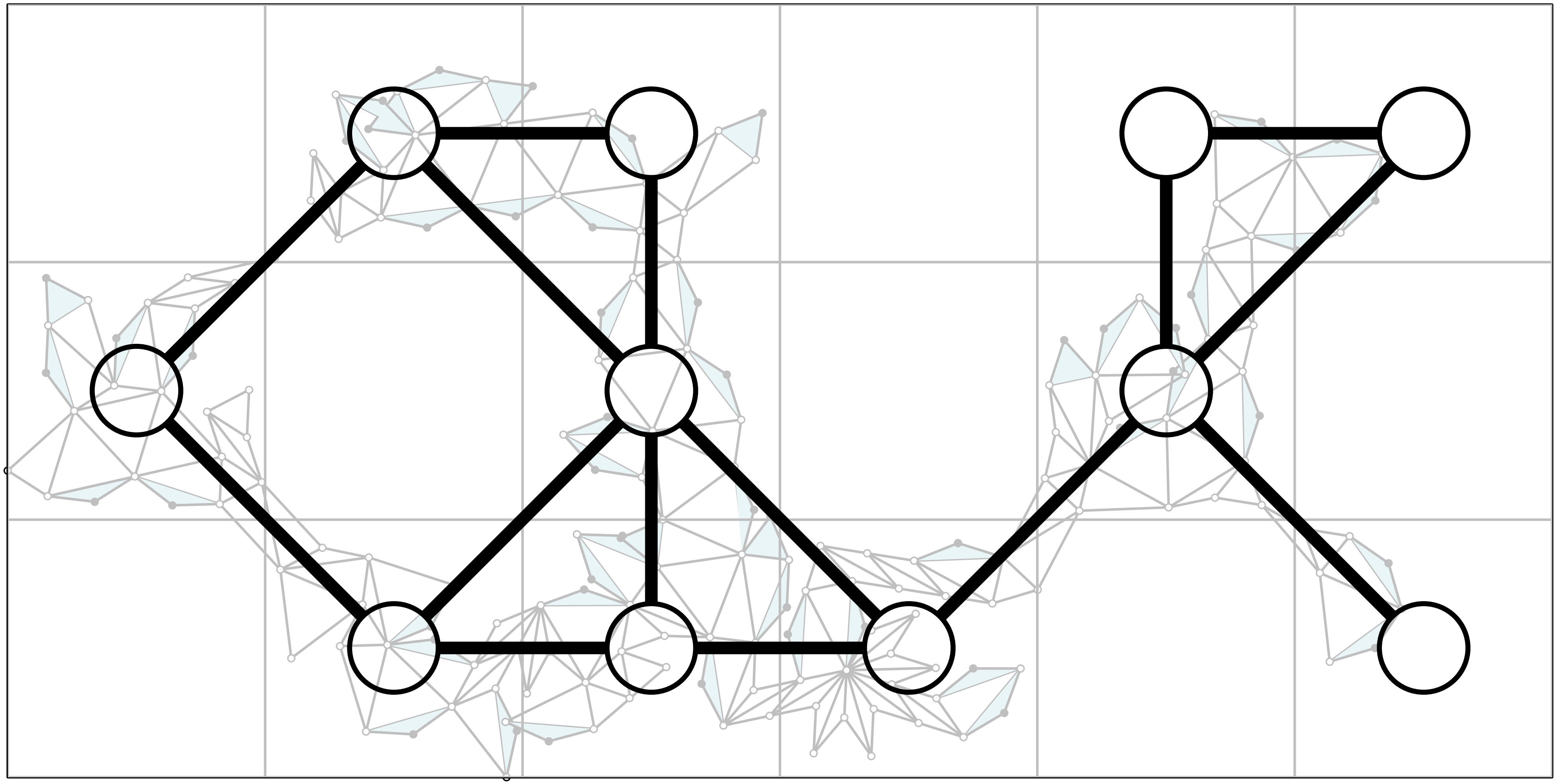


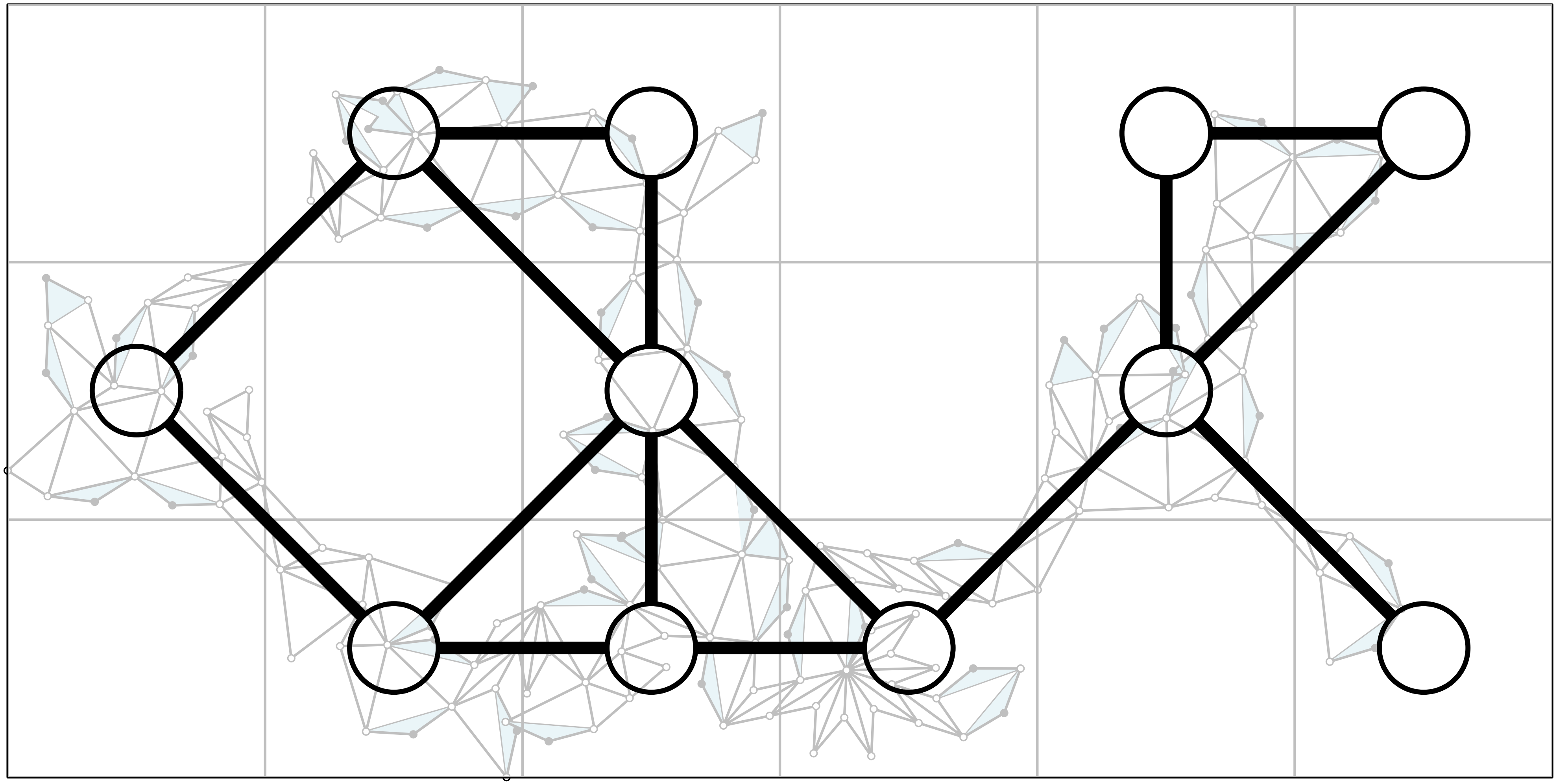












[Aspvall et al., 1979]

We can test if a 2SAT is satisfiable in linear time.

Theorem. Let $G = (V, E, \lambda)$ be an n-vertex weighted 2-tree, where $\lambda : E \rightarrow \{w_1, w_2\}$ with $w_1, w_2 \in \mathbb{R}^+$. There exists an $O(n)$-time algorithm that tests whether G admits a planar straight-line realization and, in the positive case, constructs such a realization.

Conclusions

- Fixed embedding:

Conclusions

- Fixed embedding:
  - Linear-time algorithm

Conclusions

- Fixed embedding:
  - Linear-time algorithm
- Variable embedding:

Conclusions

- Fixed embedding:
  - Linear-time algorithm
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4

Conclusions

- Fixed embedding:
    - Linear-time algorithm
- Variable embedding:
    - NP-hard if the number of distinct lengths is at least 4
    - Linear-time algorithm if the number of distinct lengths is 1 or 2

Conclusions

- Fixed embedding:
  - Linear-time algorithm
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length

Conclusions

- Fixed embedding:
  - Linear-time algorithm
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length
  - Linear-time algorithm for outerpaths

Conclusions

- Fixed embedding:
  - Linear-time algorithm
- Variable embedding:
  - NP-hard if the number of distinct lengths is at least 4
  - Linear-time algorithm if the number of distinct lengths is 1 or 2
  - Polinomial-time algorithm for 2-trees whose longest path has bounded length
  - Linear-time algorithm for outerpaths
  - Cubic-time algorithm for outerpillars

Conclusions

- Fixed embedding:
    - Linear-time algorithm
- Variable embedding:
    - NP-hard if the number of distinct lengths is at least 4
    - Linear-time algorithm if the number of distinct lengths is 1 or 2
    - Polinomial-time algorithm for 2-trees whose longest path has bounded length
    - Linear-time algorithm for outerpaths
    - Cubic-time algorithm for outerpillars

Open problems

Open problems

- What is the complexity of the problem for weighted graphs with three prescribed distinct lengths?

Open problems

- What is the complexity of the problem for weighted graphs with three prescribed distinct lengths?
- Solve the problem for general maximal outerplanar graphs.

Open problems

- What is the complexity of the problem for weighted graphs with three prescribed distinct lengths?
- Solve the problem for general maximal outerplanar graphs.
- Is there an FPT algorithm parametrized by the size of the longest path?

Thanks!